Axisymmetric Thermal Plumes

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Introduction

- Thermal plumes arise due to buoyancy forces caused by temperature differences.
- Axisymmetric plumes are common in geophysical and engineering flows.
- Goal: derive governing equations, simplify using scaling and boundary layer theory and find analytical solutions.

Mathematical Model: Momentum Balance

The governing equations for axisymmetric flow in cylindrical coordinates (r, z).

z-momentum:

$$\rho \left[v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right] - \rho g \tag{1}$$

r-momentum:

$$\rho \left[v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right] = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right] \tag{2}$$

Mathematical Model: Mass and Energy

Conservation of Mass:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{3}$$

Energy Equation:

$$v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_v} \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right] \tag{4}$$

Equilibrium State

Consider a state with no heat source where $\boldsymbol{v} = \mathbf{0}$, $T = T_0$, $p = p_0(z)$, and $\rho = \rho_0$.
Substituting these into the momentum equations yields:

$$\frac{dp_0}{dz} = -\rho_0 g \tag{5}$$

Integrating with respect to z:

$$p_0(z) = -\rho_0 g z + c_1 \tag{6}$$

Using boundary condition $p_0(H) = 0$:

$$p_0(z) = \rho_0 g(H - z) \tag{7}$$

Perturbed State

We introduce perturbations:

$$\rho = \rho_0 + \rho_1(r,z), \quad p = p_0(z) + p_1(r,z), \quad T = T_0 + S(r,z)$$

We need $\rho_1(r,z)$ in terms of $S = T - T_0$. We have that

$$\rho(T) = \rho(T_0) + \rho_1(T) \tag{8}$$

we can write

$$\rho(T) = \rho(T_0 + T - T_0) \tag{9}$$

Using Taylor's expansion:

$$\rho(T) = \rho(T_0) + \frac{d\rho}{dT}(T_0)S + O(S^2) \tag{10}$$

Defining the coefficient of thermal expansion $\beta_0 = -\frac{1}{\rho_0}\frac{d\rho}{dT}(T_0)$, we obtain:

$$\rho_1 = -\beta_0\rho_0 S \tag{11}$$

Perturbed Momentum Equations

Substituting perturbations and ρ_1 into the momentum equations:

z-momentum:

$$\rho_0 \left[v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right] = -\frac{\partial p_1}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right] + \beta_0 \rho_0 g S \tag{12}$$

r-momentum:

$$\rho_0 \left[v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right] = -\frac{\partial p_1}{\partial r} + \mu \left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right] \tag{13}$$

Energy Equation (Perturbed):

$$v_r \frac{\partial S}{\partial r} + v_z \frac{\partial S}{\partial z} = \frac{k}{\rho c_v} \left[\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + \frac{\partial^2 S}{\partial z^2} \right] \tag{14}$$

Dimensionless Variables

We introduce the following dimensionless variables:

- Characteristic length in the $z-$direction $= L$.
- Characteristic length in the $r-$direction $= \delta$.
- Characteristic velocity in the $z-$direction $= U_0$.
- Characteristic velocity in the $r-$direction $= \frac{\delta U_0}{L}$.
- Characteristic pressure$= P_c$.
- Characteristic temperature$= \Delta T$

Dimensionless Variables

Thus

- $\overline{z} = z/L$
- $\overline{r} = r/\delta$
- $\overline{v}_z = v_z/U_0$
- $\overline{v}_r = \frac{v_r L}{U_0 \delta}$
- $\overline{P}_1 = P_1/P_c$
- $\overline{S} = S/\Delta T$

Substituting these introduces characteristic scales P_c and U_0 which must be determined by scaling arguments.

Non-dimensional governing equations

Momentum equations

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{P_c}{\rho_0 U_0^2} \frac{\partial \overline{P}_1}{\partial \overline{z}} + \frac{\nu}{U_0 L} \left(\frac{L}{\delta}\right)^2 \left[\frac{\partial^2 \overline{v}_z}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{v}_z}{\partial \overline{r}} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \overline{v}_z}{\partial \overline{z}^2}\right] + \frac{g \beta_0 L \Delta T}{U_0^2} \overline{S}, \tag{15}$$

$$\left(\frac{\delta}{L}\right)^2 \left(\overline{v}_r \frac{\partial \overline{v}_r}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial \overline{z}}\right) = -\frac{P_c}{\rho_0 U_0^2} \frac{\partial \overline{P}_1}{\partial \overline{r}} + \frac{\nu_0}{U_0 L} \left[\frac{\partial^2 \overline{v}_r}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{v}_r}{\partial \overline{r}} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \overline{v}_r}{\partial \overline{z}^2} - \frac{\overline{v}_r}{\overline{r}^2}\right]. \tag{16}$$

Mass conservation

$$\frac{\partial \overline{v}_r}{\partial \overline{r}} + \frac{\overline{v}_r}{\overline{r}} + \frac{\partial \overline{v}_z}{\partial \overline{z}} = 0. \tag{17}$$

Energy equation

$$\overline{v}_r \frac{\partial \overline{S}}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{S}}{\partial \overline{z}} = \frac{k}{\rho c_v} \frac{L}{U_0 \delta^2} \left[\frac{\partial^2 \overline{S}}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{S}}{\partial \overline{r}} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \overline{S}}{\partial \overline{z}^2}\right]. \tag{18}$$

Scaling Arguments

To determine the scales, we balance the leading order terms.

1. **Pressure Scale:** Pressure gradient balances inertia/viscosity:

$$P_c = \rho U_0^2$$

2. **Boundary Layer Thickness:** For $Re \gg 1$:

$$\frac{1}{Re}\left(\frac{L}{\delta}\right)^2 = 1 \implies \left(\frac{\delta}{L}\right)^2 = \frac{1}{Re}$$

3. **Velocity Scale:** Buoyancy balances inertia:

$$\frac{L\beta_0 \Delta T}{U_0^2} = 1 \implies U_0 = (L\beta_0 \Delta T)^{1/2}$$

Boundary Layer Approximations

We assume $Re \gg 1$ and neglect terms of $O(1/Re)$ (specifically the second derivatives in z).

z-momentum:

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{\partial \overline{P}_1}{\partial \overline{z}} + \frac{\partial^2 \overline{v}_z}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{v}_z}{\partial \overline{r}} + \overline{S} \tag{19}$$

r-momentum:

$$0 = -\frac{\partial \overline{P}_1}{\partial \overline{r}} \tag{20}$$

Final Reduced System

The final equations governing the thermal plume:

Continuity:

$$\frac{\partial \overline{v}_r}{\partial \overline{r}} + \frac{\overline{v}_r}{\overline{r}} + \frac{\partial \overline{v}_z}{\partial \overline{z}} = 0 \tag{21}$$

z-momentum:

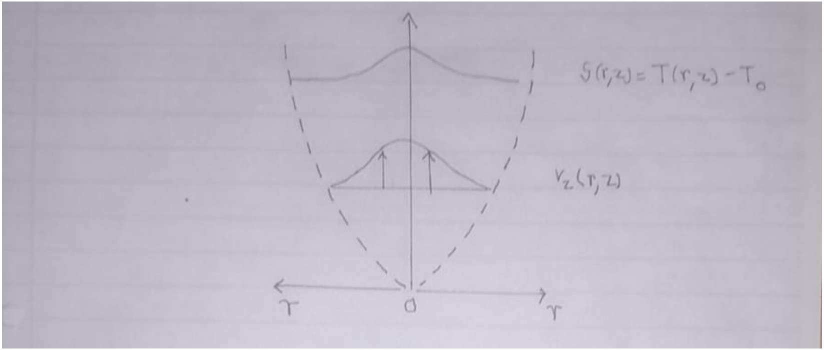
$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{\partial \overline{P}_1}{\partial \overline{z}} + \frac{\partial^2 \overline{v}_z}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{v}_z}{\partial \overline{r}} + \overline{S} \tag{22}$$

Energy:

$$\overline{v}_r \frac{\partial \overline{S}}{\partial \overline{r}} + \overline{v}_z \frac{\partial \overline{S}}{\partial \overline{z}} = \frac{1}{Pr} \left[\frac{\partial^2 \overline{S}}{\partial \overline{r}^2} + \frac{1}{\overline{r}} \frac{\partial \overline{S}}{\partial \overline{r}} \right] \tag{23}$$

Plume Diagram

Let's take a look at an axisymmetric plume diagram



Boundary Conditions

$$S(r,z) = T(r,z) - T_0$$

(i)	$r = 0:$	$v_r(0,z) = 0$	(radial velocity vanishes on axis)
(ii)	$r = 0:$	$\dfrac{\partial v_z}{\partial r}(0,z) = 0$	(local maximum of $v_z(r,z)$)
(iii)	$r = \infty:$	$v_z(\infty,z) = 0$	(no outer flow)
(iv)	$r = \infty:$	$\dfrac{\partial v_z}{\partial r}(\infty,z) = 0$	(if needed)
(v)	$r = 0:$	$\dfrac{\partial S}{\partial r}(0,z) = 0$	(local maximum)
(vi)	$r = \infty:$	$S(\infty,z) = 0$	($T(\infty,z) = T_0$)
(vii)	$r = \infty:$	$\dfrac{\partial S}{\partial r}(\infty,z) = 0$	(if needed. No radial heat flux)

Also if needed

$$\lim_{r\to\infty} r^n \frac{\partial v_z}{\partial r}(r,z) = 0, \qquad \lim_{r\to\infty} r^n \frac{\partial S}{\partial r}(r,z) = 0$$

Determining the Conserved Quantity

From the Energy equation:

$$v_r \frac{\partial S}{\partial r} + v_z \frac{\partial S}{\partial z} = \frac{1}{Pr}\left[\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r}\right] \tag{24}$$

Multiply by r and integrate from 0 to ∞:

$$\int_0^\infty r v_r \frac{\partial S}{\partial r} dr + \int_0^\infty r v_z \frac{\partial S}{\partial z} dr = \frac{1}{Pr}\int_0^\infty \frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) dr. \tag{25}$$

Using the boundary conditions at $r = 0$ and $r \to \infty$,

$$\int_0^\infty r v_r \frac{\partial S}{\partial r} dr + \int_0^\infty r v_z \frac{\partial S}{\partial z} dr = 0. \tag{26}$$

Determining the Conserved Quantity

Apply integration by parts to the first term:

$$\int_0^\infty rv_r \frac{\partial S}{\partial r} dr = -\int_0^\infty S \frac{\partial}{\partial r}(rv_r) dr. \tag{27}$$

Using the conservation of mass equation,

$$\frac{\partial}{\partial r}(rv_r) = -r\frac{\partial v_z}{\partial z}, \tag{28}$$

we obtain

$$\int_0^\infty rv_r \frac{\partial S}{\partial r} dr = \int_0^\infty rS \frac{\partial v_z}{\partial z} dr. \tag{29}$$

Thus,

$$\int_0^\infty \left[rS\frac{\partial v_z}{\partial z} + rv_z \frac{\partial S}{\partial z} \right] dr = 0. \tag{30}$$

Conserved Quantity and Flux

The integrand is a total z–derivative:

$$\int_0^\infty r\frac{\partial}{\partial z}(Sv_z)dr = 0 \quad \Rightarrow \quad \frac{d}{dz}\int_0^\infty rSv_z dr = 0. \tag{31}$$

Hence,

$$\int_0^\infty rS(r,z)v_z(r,z)dr = \text{constant independent of } z. \tag{32}$$

Thus, the plume flux is:

$$\overline{\dot{Q}} = 2\pi\int_0^\infty rS(r,z)v_z(r,z)dr. \tag{33}$$

Momentum Equation

We now consider the momentum equation. Multiplying throughout by r gives

$$rv_r\frac{\partial v_z}{\partial r} + rv_z\frac{\partial v_z}{\partial z} = \frac{\partial v_z}{\partial r} + S. \tag{34}$$

Integrating with respect to r from 0 to ∞ yields

$$\frac{d}{dz}\int_0^{\infty} rv_z^2(r,z)\,dr = \int_0^{\infty} rS(r,z)\,dr. \tag{35}$$

we plan to use this equation later

Stream Function Formulation

We introduce the axisymmetric stream function $\psi(r, z)$:

$$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{36}$$

Substitute into the continuity equation,

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0. \tag{37}$$

This gives

$$\frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{1}{r}\left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0,$$

which simplifies identically to

$$0 = 0.$$

Conservation of mass is identically satisfied.

Reduced Governing System

With continuity identically satisfied, the governing equations reduce from three equations to the following two:

Momentum equation:

$$-\frac{1}{r}\frac{\partial\psi}{\partial z}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]+\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]+S. \tag{38}$$

Energy equation:

$$-\frac{\partial\psi}{\partial z}\frac{\partial S}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial S}{\partial z}=\frac{1}{Pr}\frac{\partial}{\partial r}\left[r\frac{\partial S}{\partial r}\right]. \tag{39}$$

Similarity Solution

We introduce the following scaling transformations

$$\bar{z} = \lambda^{a} z, \quad \bar{r} = \lambda^{b} r, \quad \overline{\psi} = \lambda^{c} \psi, \quad \overline{S} = \lambda^{m} S. \tag{40}$$

Substituting into the reduced governing equations yields

$$\begin{aligned} -\frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{z}}\left[\frac{\partial}{\partial\bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{r}}\right)\right] + \frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{r}}\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{r}}\right) = \\ \lambda^{-a+c}\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left[\bar{r}\frac{\partial}{\partial\bar{r}}\left(\frac{1}{\bar{r}}\frac{\partial\overline{\psi}}{\partial\bar{r}}\right)\right] + \lambda^{-4b-a+2c-m}\overline{S} \end{aligned} \tag{41}$$

$$-\frac{\partial\overline{\psi}}{\partial\bar{z}}\frac{\partial\overline{S}}{\partial\bar{r}} + \frac{\partial\overline{\psi}}{\partial\bar{r}}\frac{\partial\overline{S}}{\partial\bar{z}} = \lambda^{c-a}\frac{1}{Pr}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\overline{S}}{\partial\bar{r}}\right) \tag{42}$$

For invariance under scaling, the exponents must satisfy

$$a = c, \qquad -4b - a + 2c - m = 0. \tag{43}$$

General Similarity Solutions

Let

$$\psi = f(z, r), \qquad S = g(z, r), \tag{44}$$

with

$$\overline{\psi} = f(\overline{z}, \overline{r}). \tag{45}$$

Scaling implies

$$\lambda^c f = f(\lambda^a z, \lambda^b r). \tag{46}$$

Differentiating with respect to λ and setting $\lambda = 1$ gives

$$cf = azf_z + brf_r. \tag{47}$$

The characteristic equations are

$$\frac{dz}{az} = \frac{dr}{br} = \frac{df}{cf}. \tag{48}$$

Solving,

$$\psi = z^{\frac{c}{a}} F\left(\frac{r}{z^{b/a}}\right), \qquad S = z^{\frac{m}{a}} G\left(\frac{r}{z^{b/a}}\right). \tag{49}$$

Determination of Similarity Exponents

Using the conserved flux

$$Q = 2\pi \int_0^\infty rSv_z \, dr, \tag{50}$$

and substituting the similarity forms, we obtain

$$Q = 2\pi z^{\frac{m}{a}+\frac{c}{a}} \int_0^\infty G(\xi)\frac{dF}{d\xi}d\xi, \quad \xi = \frac{r}{z^{b/a}}. \tag{51}$$

Since Q is independent of z,

$$\frac{m}{a} + \frac{c}{a} = 0. \tag{52}$$

Using the invariance conditions,

$$\frac{m}{a} = -1, \qquad \frac{b}{a} = \frac{1}{2}. \tag{53}$$

Thus,

$$\psi = zF(\xi), \quad S = \frac{1}{z}G(\xi), \quad \xi = \frac{r}{z^{1/2}}. \tag{54}$$

Reduction to ODEs and Solution

Substituting the similarity forms into the governing equations gives

$$\frac{1}{\xi}\frac{d}{d\xi}\left[\xi\frac{d}{d\xi}\left(\frac{1}{\xi}F'\right)\right]+\frac{1}{\xi}\frac{d}{d\xi}\left(\frac{1}{\xi}FF'\right)-\frac{1}{\xi^2}(F')^2+G=0, \tag{55}$$

$$-\frac{d}{d\xi}(FG)=\frac{1}{Pr}\frac{d}{d\xi}\left(\xi\frac{dG}{d\xi}\right). \tag{56}$$

From the integral constraint,

$$\frac{d}{dz}\int_0^\infty rv_z^2(r,z)\,dr=\int_0^\infty rS(r,z)\,dr. \tag{57}$$

Writing in terms of the similarity solutions we obtain

$$\int_0^\infty \xi\left[G(\xi)-\frac{1}{\xi^2}\left(\frac{dF}{d\xi}\right)^2\right]d\xi=0. \tag{58}$$

$$G(\xi)=\frac{1}{\xi^2}\left(\frac{dF}{d\xi}\right)^2. \tag{59}$$

Solutions for F(ξ)$and G(\xi)$

Substitution yields the following results for F and G;

$$F(\xi) = \frac{4k_1\xi^2}{k_1\xi^2 - 1}. \tag{60}$$

$$G(\xi) = \frac{64k_1}{k_1\xi^2 - 1}. \tag{61}$$

Using the conserved quantity (86), we solve for the constant α, that is

$$\alpha = \pm 16 \left(\frac{2\pi}{5\dot{\overline{Q}}} \right)^{\frac{1}{2}}, \tag{62}$$

In this case, we will choose α to be non-negative, that is, to avoid singularity.

We obtain the value for the Prandtl number for the special case of $G(\xi)$

$$Pr = 2. \tag{63}$$

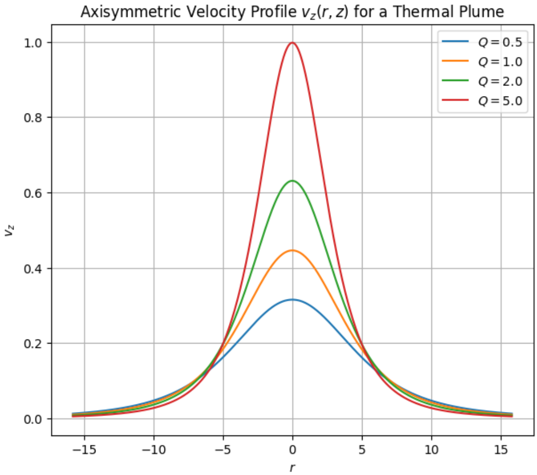
We now write v_r, v_z, and S in terms of ξ

$$v_z = \frac{8\alpha}{(\xi^2 + \alpha)^2}, \tag{64}$$

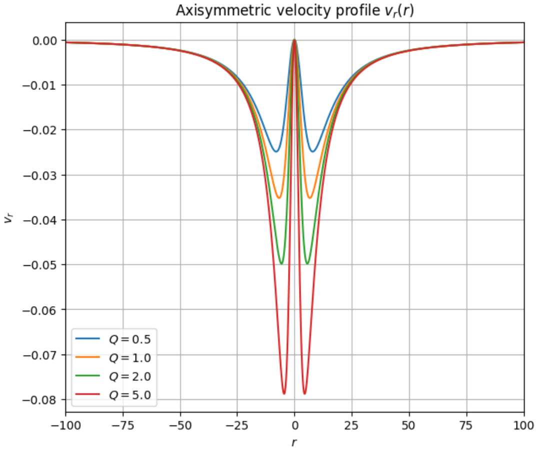
$$v_r = -\frac{4\xi^3}{z^{\frac{1}{2}}(\xi^2 + \alpha)^2}, \tag{65}$$

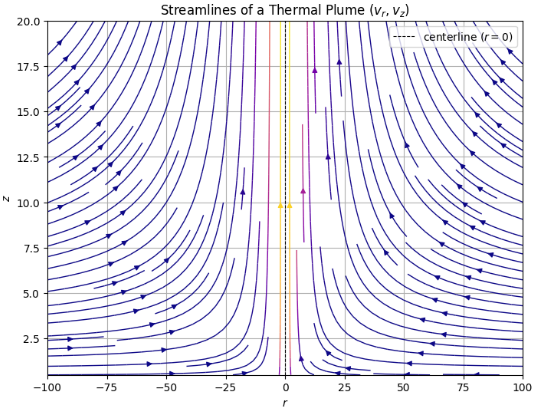
$$S = \frac{1}{z}\frac{64\ \alpha^2}{(\xi^2 + \alpha)^4}. \tag{66}$$

Results

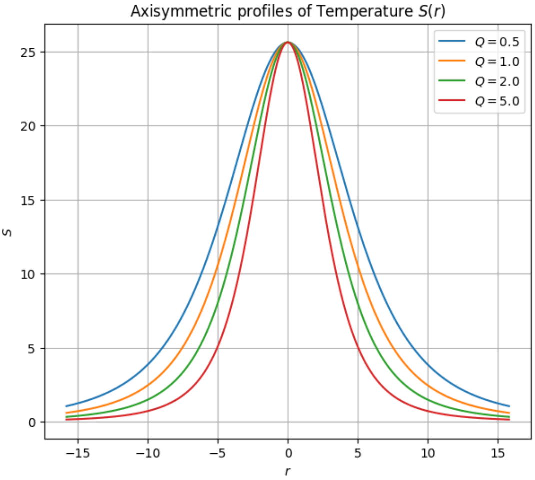


Axisymmetric Velocity Profile $v_z(r, z)$ for a Thermal Plume
$Q = 0.5$
$Q = 1.0$
$Q = 2.0$
$Q = 5.0$
1.0
0.8
0.6
0.4
0.2
0.0
v_z
−15
−10
−5
0
5
10
15
r

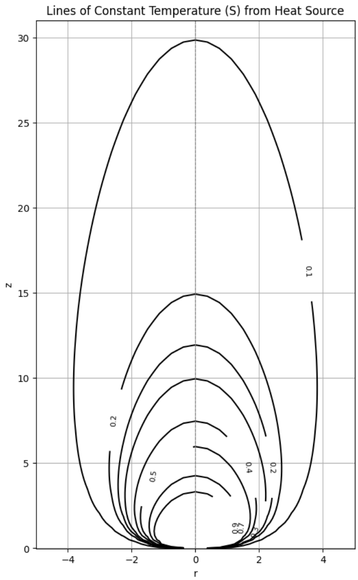

Axisymmetric velocity profile $v_r(r)$
0.00
−0.01
−0.02
−0.03
−0.04
−0.05
−0.06
−0.07
−0.08
v_r
−100
−75
−50
−25
0
25
50
75
100
r
$Q = 0.5$
$Q = 1.0$
$Q = 2.0$
$Q = 5.0$



Streamlines of a Thermal Plume (v_r, v_z)
centerline $(r = 0)$
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
z
−100
−75
−50
−25
0
25
50
75
100
r



Axisymmetric profiles of Temperature S(r)
Q = 0.5
Q = 1.0
Q = 2.0
Q = 5.0
S
r
25
20
15
10
5
0
−15
−10
−5
0
5
10
15


Lines of Constant Temperature (S) from Heat Source
z
r
0.1
0.2
0.2
0.4
0.5

Thank You!

Questions?