Understanding Tsunamis

The investigation of the Role of the KdV Equation

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January 18, 2025

Overview

1. What is a tsunami
2. Tsunamis: Formation, Coastal Impact, and Variability
3. Data
4. KdV Equation
5. One soliton solution
6. Two-Soliton Solution

What is a tsunami

- The word tsunami is Japanese for “harbour wave.” where Tsu—habour and nami—wave.
- A tsunami is a catastrophic ocean wave, usually caused by a submarine earthquake, an underwater or coastal landslide, or a volcanic eruption.
- Although often called tidal waves, the occurrence of tsunamis have no connection with tides [National Oceanic and Atmospheric Administration (NOAA), 2025].

Tsunamis: Formation, Coastal Impact, and Variability

- Tsunamis are immense ocean waves generated by earthquakes or the motion of tectonic plates, particularly along subduction zones where tectonic activity is most pronounced. These waves propagate across the ocean surface following the initial generating impulse. In deep water, tsunamis can travel at astonishing speeds of up to 800 km per hour, with extremely long wavelengths and small wave amplitudes, rendering them nearly invisible among regular wind waves and ocean swells. Their wave periods, ranging from five minutes to over an hour, further contribute to their subtlety in deep water, making them difficult to detect until they approach shallower coastal regions[National Oceanic and Atmospheric Administration (NOAA), 2025].
- As the waves approach the coast of a continent, friction with the rising sea bottom reduces the velocity of the waves, resulting in shortened wavelengths and increased wave amplitudes. Coastal waters may rise as high as 30 meters above normal sea level in 10 to 15 minutes. Between three and five major oscillations generate most of the damage, often appearing as powerful "run-ups" of rushing water that uproot trees, pull buildings off their foundations, carry boats far inshore, and wash away entire beaches, peninsulas, and other low-lying coastal formations[National Oceanic and Atmospheric Administration (NOAA), 2025, Britannica, nd].
- Tsunamis are reflected and refracted by the topography of the seafloor near shore and the configuration of a coastline, resulting in their effects varying widely from place to place. A scale, Richter scale, is used to describe the magnitude of the tsunami. This ranges from 0—little to no waves, to 10—waves with enormous amplitude at the shore [National Oceanic and Atmospheric Administration (NOAA), 2025]

TSUNAMI: Earthquake

Ocean(ocean depth in meters)	Richter Scale reading	wave height(meters)	Casualties
Eastern Mediterranean Sea (5,267)	8.0–8.5	10	$>$ 10 000
Indian (3,741)	9.1	51	227,899
Pacific (4,280)	9.0-9.1	40	18,453
Pacific (4,280)	7.6-7.9	38.2	27,122

Table: Ref:[National Oceanic and Atmospheric Administration (NOAA), 2025]

TSUNAMI: Volcano

Ocean(ocean depth in meters)	Richter Scale reading	wave height(meters)	Casualties
Pacific (4,280)	7.0	57	14,524
Indian (3,741)	4.8-5.9	42	34,417

Table: Ref:[National Oceanic and Atmospheric Administration (NOAA), 2025]

Data

Ocean	Topography	Places Frequently Affected
Pacific Ocean	- Known as the "Ring of Fire" due to high tectonic activity. - Deep trenches (e.g., Mariana Trench) and subduction zones. - Numerous underwater volcanoes and fault lines.	- Japan - Indonesia - Chile - Alaska (USA)
Indian Ocean	- Less active than the Pacific but has significant subduction zones (e.g., Sunda Trench). - Relatively shallow continental shelves near coastlines. - Limited seismic monitoring historically, making tsunamis more deadly.	- India - Sri Lanka - Indonesia (Sumatra)
Atlantic Ocean	- Less tectonically active but features the Mid-Atlantic Ridge. - Steep slopes along the continental margins can cause underwater landslides. - Rare subduction zones compared to the Pacific and Indian Oceans.	- Portugal (Lisbon area) - Caribbean nations (e.g., Puerto Rico, Dominican Republic) - Eastern Canada (Newfoundland)

KdV Equation

- The Korteweg–de Vries (KdV) equation is a nonlinear partial differential equation.
- It describes the propagation of long, one-dimensional waves in shallow water with weakly nonlinear restoring forces and weak dispersion.
- The KdV equation is widely used in physics and applied mathematics to model various wave phenomena, including:
  - Water waves
  - Plasma waves
  - Acoustic waves in a crystal lattice
- The equation balances two key effects:
  - Nonlinear effects that tend to steepen the wave.
  - Dispersive effects that spread the wave out.
- This balance allows for the formation of solitons:
  - Solitons are stable, localized waves that maintain their shape as they travel.

KdV Equation

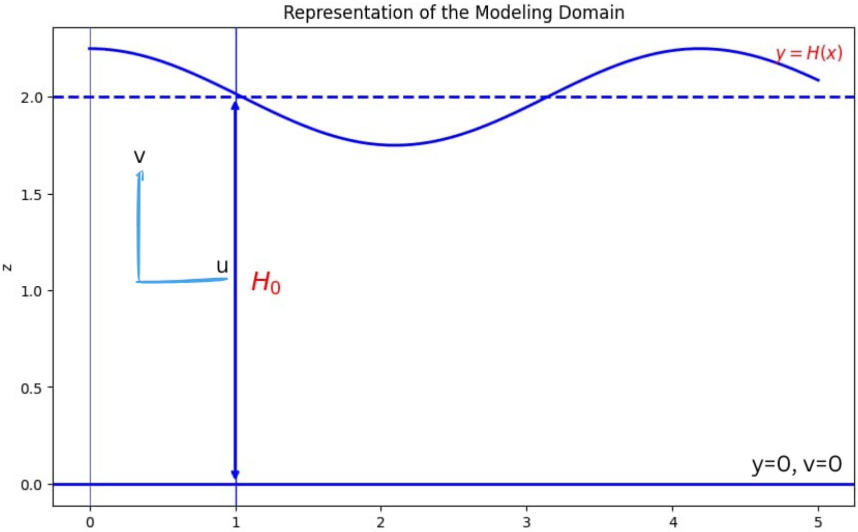
The KdV Equation The standard form of the KdV equation is:

$$\eta_t + \alpha\eta\eta_x + \eta_{xxx} = 0 \tag{1}$$

Where:

- $\eta(x, t)$: Represents the wave profile or amplitude at position x and time t.
- η_t: Describes the time evolution of the wave.
- $\alpha\eta\eta_x$: Represents the nonlinear effect, where the wave speed depends on the wave amplitude.
- η_{xxx} : Accounts for the dispersive effects, which cause the wave to spread.

Schematic Representation of Ocean Body Waves in KdV Equation Derivation



The variables of ocean body waves

- $u(x, y)$: is speed in the $+ve$ X direction.
- $v(x, y)$: is speed in the $+ve$ Y direction.
- $\underline{\varphi} = \varphi(u, v)$
- H_0: height of the wave at rest.
- $H(x)$: height of the wave at x.

Reduction of Navier–Stokes Equations to Euler's Equation

The Navier–Stokes equations for a Newtonian fluid are:

$$\frac{\partial \underline{\varphi}}{\partial t} + (\underline{\varphi} \cdot \nabla)\underline{\varphi} = -\frac{1}{\rho}\nabla p + \mu \nabla^2 \underline{\varphi} + \underline{F}, \tag{2}$$

where:

- $\underline{\varphi}$ is the velocity vector field,
- t is time,
- p is pressure,
- ρ is the fluid density,
- μ is the kinematic viscosity,
- $\underline{F}$ represents external forces per unit volume.

The continuity equation for incompressible flow is:

$$\nabla \cdot \underline{\varphi} = 0. \tag{3}$$

Reduction of Navier–Stokes Equations to Euler's Equation

- Constant pressure at the surface.
- Inviscid.
- Surface-tension effects negligible.
- frictional forces are negligible.
- neglect viscosity.
- irrotational flow.

Reduction of Navier–Stokes Equations to Euler's Equation

Assume the model satisfies the above conditions, then;

$$\frac{\partial \underline{\varphi}}{\partial t} + (\underline{\varphi} \cdot \nabla)\underline{\varphi} = -\frac{1}{\rho}\nabla p + \underline{F}, \tag{4}$$

Reduction of Navier–Stokes Equations to Euler's Equation

The components:

$$u_t + uu_x + vu_y = -\frac{1}{\rho} p_x \qquad (5)$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho} P_y - g \qquad (6)$$

$$u_x + v_y = 0 \qquad \text{incompressible} \qquad (7)$$

$$v_x - u_y = 0 \qquad \text{irrotational} \qquad (8)$$

Boundary conditions:

$$v = 0 \quad \text{at} \quad y = 0 \qquad (9)$$

$$P = P_{atm} \quad \text{at} \quad y = H(x) \qquad (10)$$

Non-dimensionalisation of the Euler' equation

$$
\begin{aligned}
u &= \epsilon\sqrt{gH_0}u^* && (11)\\
v &= \frac{\epsilon}{\lambda}\sqrt{gH_0}v^* && (12)\\
x &= \lambda x^* && (13)\\
y &= H_0 y^* && (14)\\
t &= \frac{\lambda}{\sqrt{gH_0}}t^* && (15)\\
p &= \epsilon\rho g H_0 p^* && (16)
\end{aligned}
$$

where

- $\sqrt{gH_0}$ is the scaling with regard to the wave's speed.
- $\epsilon = \frac{a}{H_0}$ is the scaling parameter , where a is the amplitude of the wave.

Non-dimensionalisation of the Euler's equation

substituting the new scaled variables in the Euler's equations we get the follows:

$$
\begin{aligned}
u_t^* + \epsilon(u^* u_x^* + v^* u_y^*) &= -p_x^* && (17)\\
\delta^2(v_t^* + \epsilon(u^* v_x^* + v^* v_y^*) &= -p_y^* && (18)\\
u_x^* + v_y^* &= 0 && (19)\\
\delta^2 v_x^* - u_y^* &= 0 && (20)
\end{aligned}
$$

Non-dimensionalisation of the Euler's equation

Introduce the velocity potentials given
by [Constantin and Henry, 2009, Constantin and Johnson, 2008]:

$$\epsilon v^* = \frac{\partial \Phi}{\partial x^*} \tag{21}$$

$$u^* = \frac{\partial \Phi}{\partial y^*} \tag{22}$$

substituting (21) and (22) into non-dimensional Euler's equation, we get

$$\epsilon \Phi_{xx} + \Phi_{yy} = 0 \quad \text{in} \quad 0 \leq y \leq 1 + \epsilon\eta \tag{23}$$

$$\Phi_y = 0 \quad \text{in} \quad y = 0 \tag{24}$$

$$\Phi_y = \epsilon(\eta_t + \epsilon \Phi_x \eta_x) \quad \text{in} \quad y = 1 + \epsilon\eta \tag{25}$$

$$\Phi_t + \frac{\epsilon}{2}\Phi_x^2 + \eta = 0 \quad \text{in} \quad y = 1 + \epsilon\eta \tag{26}$$

Non-dimensionalisation of the Euler's equation

Introduce perturbation, ie

$$\Phi \sim \sum_{k=0}^{\infty} \epsilon^k \Phi_k \tag{27}$$

$$\eta \sim \sum_{k=0}^{\infty} \epsilon^k \eta_k \tag{28}$$

The leading order approximation (ϵ^0) to (23) is

$$\epsilon\Phi_{xx} \sim \epsilon(\epsilon^0\Phi_{0,xx} + \epsilon^1\Phi_{1,xx} + \dots) \tag{29}$$

$$\sim \epsilon\Phi_{0,xx} + \epsilon^2\Phi_{1,xx} + \dots \tag{30}$$

For (ϵ^0), $\epsilon\Phi_{xx} \sim 0$

$$\Phi_{yy} \sim \epsilon^0\Phi_{0,yy} + \epsilon^1\Phi_{1,yy} + \dots \tag{31}$$

For (ϵ^0), $\epsilon^0\Phi_{yy} \sim \Phi_{0,yy}$.

Non-dimensionalisation of the Euler's equation

Sub $\epsilon\Phi_{xx}$ and Φ_{yy} into (23) we get

$$\Phi_{0,yy} = 0 \quad \text{in} \quad 0 \leq y \leq 1 \tag{32}$$

$$\Phi_{0,y} = 0 \quad \text{in} \quad y = 1 \tag{33}$$

$$\Phi_{0,y} = 0 \quad \text{in} \quad y = 0 \tag{34}$$

$$\Phi_{0,t} + \frac{1}{2}(\Phi_{0,y})^2 + \eta_0 = 0 \quad \text{in} \quad y = 1 \tag{35}$$

- $\Phi_0(x^*, y^*, t^*)$ is independent of y^*, thus $\Phi_0 = F(t^*, x^*)$.
- As such, (35) becomes

$$F_t + \eta_0 = 0 \text{ in } y = 1. \tag{36}$$

Non-dimensionalisation of the Euler's equation

Introduce perturbation, ie

$$\Phi \sim \sum_{k=0}^{\infty} \epsilon^k \Phi_k \quad (37)$$

$$\eta \sim \sum_{k=0}^{\infty} \epsilon^k \eta_k \quad (38)$$

The leading order approximation (ϵ^1) to (32) is

$$\epsilon \Phi_{xx} \sim \epsilon(\epsilon^0 \Phi_{0,xx} + \epsilon^1 \Phi_{1,xx} + \dots) \quad (39)$$

$$\sim \epsilon \Phi_{0,xx} + \epsilon^2 \Phi_{1,xx} + \dots \quad (40)$$

For (ϵ^1), $\epsilon \Phi_{xx} \sim \Phi_{0,xx}$

$$\Phi_{yy} \sim \epsilon^0 \Phi_{0,yy} + \epsilon^1 \Phi_{1,yy} + \dots \quad (41)$$

For (ϵ^1), $\epsilon^0 \Phi_{yy} \sim \Phi_{1,yy}$.

Non-dimensionalisation of the Euler's equation

Sub $\epsilon\Phi_{xx}$ and Φ_{yy} into (23) we get

$$\Phi_{0,xx} + \Phi_{0,yy} = 0 \quad \text{in} \quad 0 \leq y \leq 1 \tag{42}$$

$$\Phi_{0,y} + \eta_{0,t} = 0 \quad \text{in} \quad y = 1 \tag{43}$$

$$\Phi_{1,y} = 0 \quad \text{in} \quad y = 0 \tag{44}$$

$$\Phi_{1,t} + \frac{1}{2}(\Phi_{0,x})^2 + \eta_1 + \Phi_{1,y}\Phi_{0,y} = 0 \quad \text{in} \quad y = 1 \tag{45}$$

- For $\Phi_0 = F(t^*, x^*)$, (42) and (44), we deduce
-
$$\Phi_{1,y} = -zF_{xx} \text{ for } 0 \leq y \leq 1. \tag{46}$$
- furthermore, (43) becomes
$$F_{xx} = -\eta_{0,t} \text{ at } y = 1 \tag{47}$$

Non-dimensionalisation of the Euler's equation

From (36), the linear wave equation produces

$$\eta_{0,xx} - \eta_{0,tt} = 0 \tag{48}$$

Process

- The general solution of the wave equation is

$$\eta_0(x,t) = f(x-t) + f(x+t) \tag{49}$$

- Introduce new non-dimensional variables such that

$$\xi = x - t \text{ and } \tau = \epsilon t \tag{50}$$

- Transform the equations (23) - (26) with respect to ξ and τ into

$$\epsilon\Phi_{\xi\xi} + \Phi_{yy} = 0 \qquad \text{in } 0 \leq y \leq 1 + \epsilon\eta \tag{51}$$

$$\Phi_y = \epsilon(-\eta_\xi + \epsilon\eta_0 + \epsilon\phi_\xi\eta_\xi) \qquad \text{at } y = 1 + \epsilon\eta \tag{52}$$

$$\Phi_y = 0 \qquad \text{at } y = 0 \tag{53}$$

$$\epsilon\Phi_\tau - \Phi_\xi + \frac{\epsilon}{2}\Phi_\xi^2 + \frac{1}{2}\Phi_y^2 + \eta = 0 \qquad \text{at } y = 1 + \epsilon\eta \tag{54}$$

Expand ϕ nad η in power of ϵ where the leading approximate ϵ^0 from (51) to (54) From equations of (ϵ^0) nad

$$\phi_0(\xi,\tau) = L(\xi,\tau)$$

The combination of equations for $(\epsilon^0), \epsilon^1)$ and $\epsilon^2)$ deduce to the following equations

$$\phi_{1,y} = -yL_{\xi\xi}(\xi,\tau), \quad in \quad 0 \leq y \leq 1. \tag{55}$$

$$\phi_1 = -\frac{1}{2}L_{\xi\xi} + B(\xi,\tau)y + q(\xi,\tau) \tag{56}$$

Using the condition $y = 1$ for $equ(c)\mathrm{B}(\xi,\tau) = 0$ equating and manipulating the $\epsilon^0, \epsilon^1, \epsilon^2$ equations results in

$$0 = \eta_{0,t} + \eta_0\eta_{0,t} + \frac{1}{6}\eta_{0,\xi\xi\xi} \tag{57}$$

(57) is the kdv equation in the transform coordinates ξ and τ

Euler's Equations with rotation

$$u_t + uu_x + v_y + 2\omega v = -\frac{1}{\rho}p_x \tag{58}$$

$$v_t + vv_x + uv_y - 2\omega u = -\frac{1}{\rho}P_y - g \tag{59}$$

$$u_x + v_y = 0 \tag{60}$$

boundary conditions

$$v = \eta_t + u\eta_x \quad or \quad y = h_0 + (x, t) \tag{61}$$

$$P = P_{atm} \quad or \quad y = h_0 + (x, t) \tag{62}$$

$$v = 0 \quad or \quad y = 0 \tag{63}$$

where $\omega = 7.29 \times 10^{-5} rads^{-1}$ denotes the rotational speed of the Earth around the polar axis, thus the two terms with ω in the equation represent the Coriolis force. $g = 9.8ms^{-2}$ We denote by p_{atm} the atmospheric pressure.

Generalized kdv

$$2\eta_t - 2\omega_0\eta_\xi + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} = 0,$$

ω_0 is the constant rotated to the Coriolis effect. Solution of the kdv equation

$$\varphi(\xi - c\tau) = 2(c + \omega_0)\text{sech}^2\left(\sqrt{\frac{3}{2}(c + \omega_0)}(\xi - c\tau)\right), \quad c > 0.$$

. ω_0 alters the shape of the solution just as the speed c does.

. if ω_0 increases, the solitary wave becomes taller and narrower.

. the effect of Earth rotation(Coriolis pressure) does not alter the quantitative feature but modifies slightly the quantitative aspects.

Limitations of the Korteweg–de Vries (KdV) Model in Tsunami Propagation

- **Absence of Frictional Forces:** The KdV equation does not account for frictional forces, such as those exerted by the ocean floor or air resistance, which are important in real-world tsunami propagation.
- **Neglect of Earth's Rotation:** The model ignores the Coriolis effect due to the Earth's rotation, which can influence large-scale wave dynamics and ocean currents, particularly in higher latitudes.
- **Assumption of a Flat Ocean Floor:** The KdV equation assumes a flat ocean base, whereas the real seafloor has complex topography (continental shelves, ridges, valleys) that can significantly affect wave behavior.
- **Exclusion of the Forcing Term:** The KdV model neglects the external forcing term, such as atmospheric pressure changes or underwater seismic activity, which play a crucial role in tsunami generation.
- **Breakdown of Non-linearity and Dispersion Near Shoreline:** As the tsunami approaches the shoreline, the balance between nonlinearity and dispersion in the KdV model breaks down. In reality, both effects become more pronounced, leading to

Waves

Waves

One soliton solution

To verify that the numerical solution is the solution, we plot both for a particular value of t [Regina and Mohamed, 2023]:

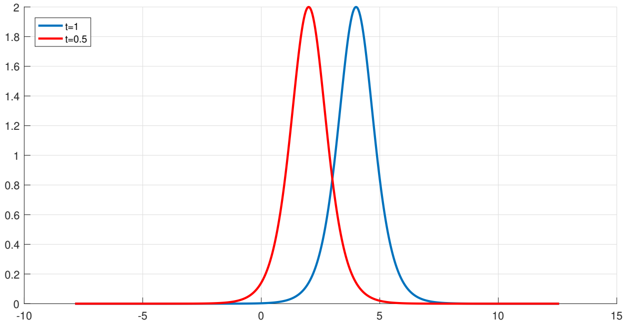


Figure: Solitary with difference t's and amplitude is two

One soliton solution

In fact, there is a whole family of 1-soliton solutions parameterized by the amplitude of the trough. These are:

$$u(x,t) = x_{max} sech^2 \left(\sqrt{\frac{x_{max}}{2}} (x - 2x_{max} t) \right), \tag{64}$$

so the higher the trough, the faster the soliton moves, and the narrower it is. We verify that this does satisfy the KdV equation:

Figure: Animation of different amplitude

Two-Soliton Solution

The theory states that an initial state:

$$u(x,0) = n(n+1)\,\mathrm{sech}^2(x), \tag{65}$$

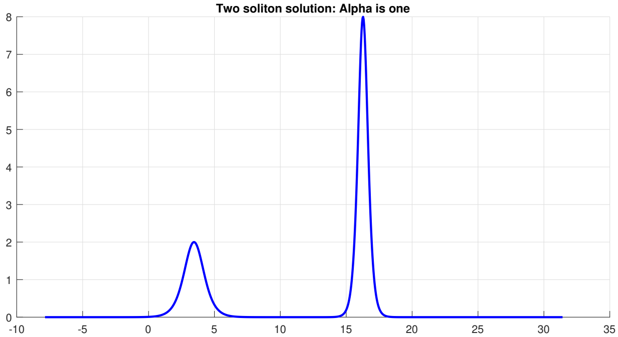
results in n solitons that propagate with different velocities. The solution for $n = 2$ is:

$$u(x,t) = 12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}. \tag{66}$$

It is not immediately evident that the above expression for $u(x,t)$ satisfies the KdV equation, but *MATLAB* confirms that it does.

Two-Soliton Solution

Next, we plot the solution at time $t = 1$:



We see a hight of amplitude 8 and a hight of amplitude 2. To determine the speeds of these hightes, we locate the minima of the function at two different times, $t = 2$ and $t = 3$.

Two-Soliton Solution

Figure: animation of the two solitons crossing each other.

Conclusion

- Tsunamis are among the most destructive natural disasters, causing widespread devastation and loss of life, particularly in coastal regions.
- Effective modeling of tsunami behavior is crucial for:
  - Early warning systems
  - Disaster preparedness
  - Mitigating impacts on affected populations
- The Korteweg–de Vries (KdV) equation has been widely used to model tsunami behavior, offering a foundational understanding of wave propagation.
- Despite its utility, the KdV equation has notable limitations:
  - Omits crucial factors such as frictional forces, Earth's rotation, ocean topography, and the influence of currents and tides.
  - Fails to maintain the balance between nonlinearity and dispersion near the shoreline, where tsunami dynamics become more complex.
- The KdV model is insufficient for accurately predicting the full scope of a tsunami's impact, particularly near the coast.
- More comprehensive models are needed to improve predictions and reduce the loss of life and property in future tsunami events.
- Continued research and development of advanced tsunami models are essential for safeguarding coastal communities.

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