Thermal Plume in Lake Kivu

Group 6

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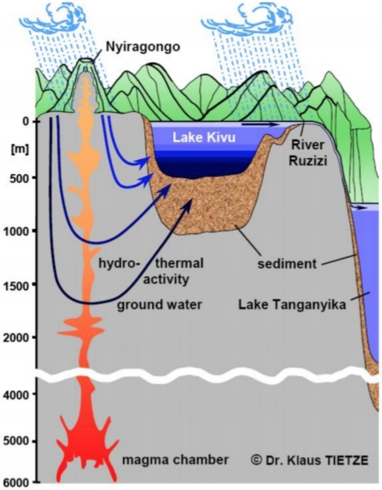
Introduction

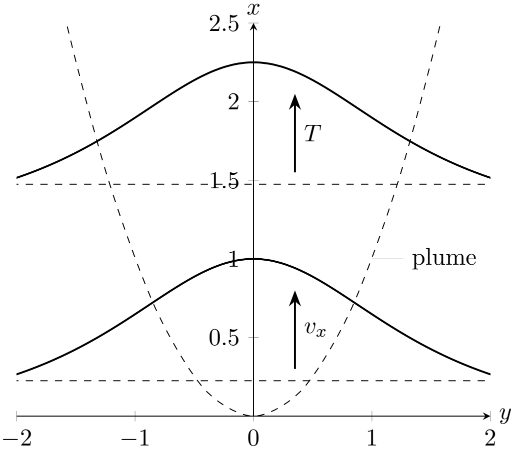
From Kay Honkanen/Ostman Agency

Two-dimensional thermal plume



Assumptions

1. Two-dimensional and laminar
2. Steady state
3. No bubbles



Governing equations

1. Conservation of mass:

$$\frac{Dp}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

2. Navier-Stokes:

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla p + \mu \nabla^2 \mathbf{v}$$

3. Energy equation:

$$\rho C_v \left[\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T \right] = k \nabla^2 T$$

Governing equations

The governing equations and assumptions leads to the model of the plume:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial S}{\partial x} + v_y \frac{\partial S}{\partial y} = \frac{k_0}{\rho_0 C_v} \left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right]$$

$$\rho_0 \left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \right] = \frac{\partial p_1}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right] + \rho_0 \beta_0 g S(x,y)$$

$$\rho_0 \left[v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} \right] = \frac{\partial p_1}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right]$$

Analysis

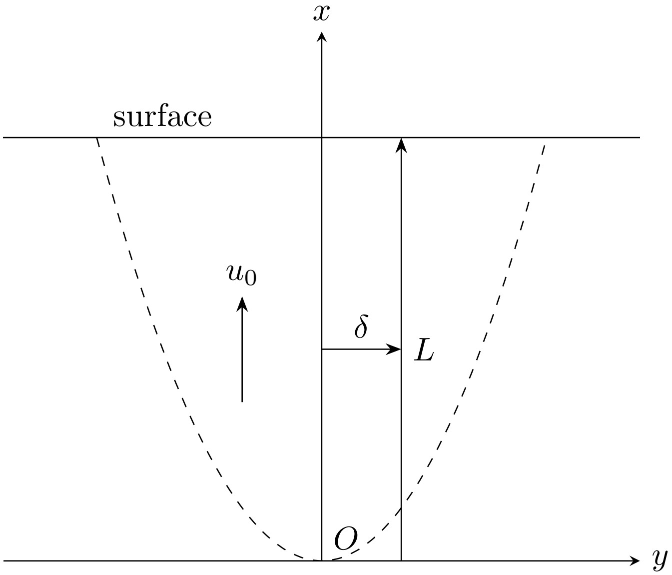
Characteristic quantities:

$$x : L = H, \quad y : \delta, \quad S : \triangle T, \quad V_x : U_0, \quad V_y : \frac{\delta U_0,}{L}, \quad P_1 : P.$$

Dimensionless variables:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}, \quad \bar{S} = \frac{S}{\triangle T}, \quad \bar{V_x} = \frac{V_x}{u_0}, \quad \bar{V_y} = \frac{L}{\delta}\frac{V_y}{u_0}, \quad \bar{p_1} = \frac{p_1}{P}.$$

Thermal plume with characteristic quantities



Summary of non-dimensionalized equations

$$\bar{V_x}\frac{\partial \bar{V_x}}{\partial \bar{x}} + \bar{V_y}\frac{\partial \bar{V_x}}{\partial \bar{y}} = -\frac{\bar{p_1}}{\bar{x}} + \left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \bar{V_x}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{V_x}}{\partial \bar{y}^2} + \bar{S}$$

$$\left(\frac{\delta}{L}\right)^2 \left[\bar{V_x}\frac{\partial \bar{V_y}}{\partial \bar{x}} + \bar{V_y}\frac{\partial \bar{V_y}}{\partial \bar{y}}\right] = -\frac{\bar{p_1}}{\bar{y}} + \left(\frac{\delta}{L}\right)^2 \left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \bar{V_y}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{V_y}}{\partial \bar{y}^2}\right]$$

$$\frac{\partial \bar{V_x}}{\partial \bar{x}} + \frac{\partial \bar{V_y}}{\partial \bar{y}} = 0$$

$$\bar{V_x}\frac{\partial \bar{S}}{\partial \bar{x}} + \bar{V_y}\frac{\partial \bar{S}}{\partial \bar{y}} = \frac{1}{P_r}\left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \bar{S}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{S}}{\partial \bar{y}^2}\right]$$

Stream function

Using the identities for the stream function,

$$\begin{aligned} V_x &= \frac{\partial \psi}{\partial y}, \\ V_y &= -\frac{\partial \psi}{\partial x}, \end{aligned}$$

the equations were then rewritten in terms of ψ:

$$\begin{aligned} \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^3 \psi}{\partial y^3} + S(x,y), \\ \frac{\partial \psi}{\partial y}\frac{\partial S}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial S}{\partial y} &= \frac{1}{P_r}\frac{\partial^2 S}{\partial y^2}. \end{aligned}$$

Scaling transformations

The scaling transformations:

$$\bar{x} = \lambda^a x, \quad \bar{y} = \lambda^b y, \quad \bar{\phi} = \lambda^c \phi, \quad \bar{S} = \lambda^m S.$$

This produced similarity equations with a similarity solution of

$$c = a - b,$$

$$m = a - 4b.$$

Let the solution:

$$\psi = f(x, y),$$
$$S = g(x, y).$$

Then

$$\bar{\psi} = f(\bar{x}, \bar{y}),$$
$$\bar{S} = g(\bar{x}, \bar{y}).$$

Continued..

Now

$$\lambda^c f(x,y) = f(\lambda^a x, \lambda^b y).$$

Differentiating with respect to λ and setting $\lambda = 1$ produces a first order quasi-linear PDE

$$cf = a\frac{\partial f}{\partial x}x + b\frac{\partial f}{\partial y}y.$$

The differential equations of the characteristic curves are then

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{df}{cf},$$

with the solutions:

$$C_1 = \frac{y}{x^{\frac{b}{a}}},$$

and

$$C_2 = \frac{f}{x^{\frac{c}{a}}}.$$

Scaling transformation solution

The general form of the similarity solution is:

$$\psi(x, y) = x^{1-\alpha} F(\xi),$$

$$S(x, y) = x^{1-4\alpha} G(\xi),$$

with

$$\xi = \frac{y}{x^{\alpha}},$$

$$\alpha = \frac{b}{a}.$$

Transforming from PDE to ODE

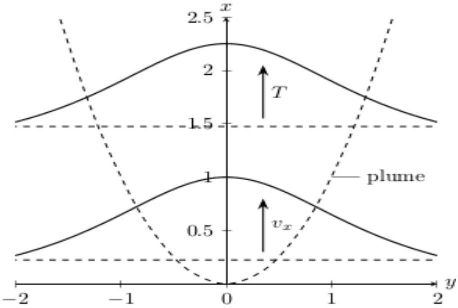
Using boundary conditions, the PDEs are converted into ODEs

$$\frac{d^3F}{d\xi^3} + (1-\alpha)\frac{d}{d\xi}\left(\frac{dF}{d\xi}\right) + (3\alpha-2)\left(\frac{dF}{d\xi}\right)^2 + G(\xi) = 0,$$
$$\frac{d^2G}{d\xi^3} + P_r\left[(1-\alpha)F\frac{dG}{d\xi} - (1-4\alpha)G\frac{dF}{d\xi}\right] = 0.$$

Boundary Conditions

For $y = \pm\infty$, $V_x(x, \pm\infty) = 0$; $\dfrac{\partial V_x(x, \pm\infty)}{\partial y} = 0$, $\dfrac{\partial S(x, \pm\infty)}{\partial y} = 0$;

and $y = 0$, $\dfrac{\partial V_x(x, 0)}{\partial y} = 0$, $V_y(x, 0) = 0$, $\dfrac{\partial S(x, 0)}{\partial y} = 0$, $S(x, \pm\infty) = 0$.



Conserved quantity

The conserved quantity calculation

$$\int_{-\infty}^{\infty} V_x(x,y)S(x,y)dy = \dot{Q}_b,$$
$$\frac{d}{dx}\int_{-\infty}^{\infty} V_x^2(x,y)dy = \frac{d}{dx}\int_{-\infty}^{\infty} S(x,y)dy,$$

where $\dot{Q}_b$ is the power per unit length of the source.

The ODEs

Using $\alpha = \frac{2}{5}$, the ODEs becomes:

$$\frac{d^3F}{d\xi^3} + \frac{3}{5}\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) = 0,$$

$$\frac{d^3F}{d\xi^3} + \left(\frac{3}{5} + b\right)\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) = 0.$$

Solutions

Normal solution:

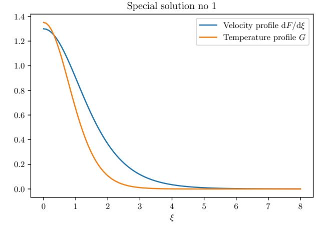
$$
\begin{aligned}
Pr &= 2, \\
G(\xi) &= \frac{4}{5}\left(\frac{dF}{d\xi}\right)^2 \\
F(\xi) &= \left(\frac{10B}{3}\right)^{\frac{1}{2}} tanh\left[\left(\frac{3B}{10}\right)^{\frac{1}{2}}\xi\right], \\
B &= \frac{1}{4}\left(\frac{15}{2}\right)^{\frac{3}{5}} \dot{Q_b}^{\frac{2}{5}}, \\
\xi &= \frac{y}{x^{\frac{2}{5}}}.
\end{aligned}
$$

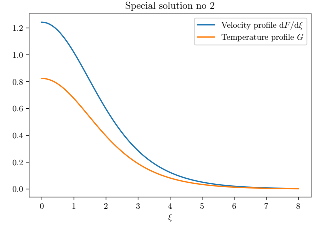
Solutions

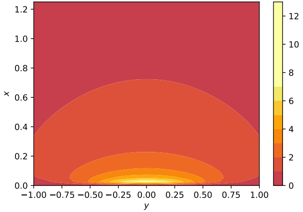
Special case solution:

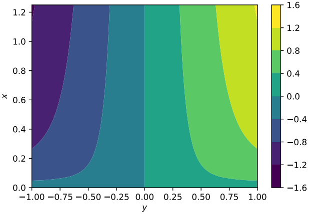
$$
\begin{aligned}
b &= -4/5, \\
Pr &= 5/9, \\
G(\xi) &= \frac{4}{5}\left(\frac{dF}{d\xi}\right)^2 + b\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right), \\
F(\xi) &= (6B)^{\frac{1}{2}} tanh\left[\left(\frac{B}{6}\right)^{\frac{1}{2}}\xi\right], \\
B &= \frac{1}{4}\left(\frac{675}{2}\right)^{\frac{1}{5}}\dot{Q_b}^{\frac{2}{5}}, \\
\xi &= \frac{y}{x^{\frac{2}{5}}}.
\end{aligned}
$$

Results


Special solution no 1
Velocity profile $\mathrm{d}F/\mathrm{d}\xi$
Temperature profile G
ξ


Special solution no 2
Velocity profile $\mathrm{d}F/\mathrm{d}\xi$
Temperature profile G
ξ


x
y


x
y

Conclusion

- In conclusion we formulated a problem, that described a laminar thermal plume in two dimensions in lake Kivu, using Navier-Stokes equations, conservation of mass and the energy equation.
- This was solved and the exact solutions were obtained that describes the boundary and flow of the plume under certain boundary conditions.
- Further work could include the bubbles.