DIFFUSER TRACER TEST INTERPRETATION

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Abstract

The modern method for removing sugar from cane is to use a reverse flow diffuser. Sugar is pulped into a fibrous bed that is carried along the length of the diffuser on a conveyor. Water is drained through the bed to flush out the sugar. The water is first introduced at the most downstream end of the diffuser, collected in trays and then reintroduced further upstream using spray jets, collected again and then moved further upstream. The process is repeated multiple times.

The process works well provided the flow from the sprays to the collection trays is consistent and predictable and the water level in the diffuser remains close to optimal. However, the diffuser unit is completely enclosed and it is difficult to determine whether these conditions are being satisfied. Experiments have been conducted by injecting a pulse of saline solution into the spray jets and collecting it in the appropriate collection trays. The concentration of salt in the collection trays can be determined using measurements of conductivity. The Study Group proposed and solved several models to try to interpret the traces of salinity obtained in these experiments. Several possible interpretations emerged and it is likely that further work will be required to remove this ambiguity.

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1 Introduction

The modern method for removing sugar from cane is to use a reverse flow diffuser. Sugar is pulped into a fibrous bed that is carried along the length of the diffuser on a conveyor. At the same time water is drained through the bed to flush out the sugar. The water is first introduced at the most downstream end of the diffuser, collected in trays and then reintroduced further upstream using spray jets, collected again and then moved further upstream. The diffusers under consideration here have around 12 trays and sprays, so that the water/sugar mixture is recycled through the bed 12 times in ideal conditions. The reason for the reverse flow is to maximize the difference between the amount of sugar in the cane and the sugar in the water/juice. Thus when first introduced, the water is fresh while the cane has little sugar left, whereas when the water makes its last pass at the upstream end the cane is rich with sugar and the juice will contain a significant amount of sugar. In previous Study Groups [1, 2] similar flow problems have been considered.

The process works well in general provided the flow from the sprays to the collection trays is consistent and predictable and the water level in the diffuser remains close to optimal (very close to the surface of the cane without flooding). However, the diffuser unit is completely enclosed and it is difficult to determine whether these conditions are being satisfied. Experiments have been conducted by injecting a pulse of saline solution into the spray jets and collecting it in the appropriate collection trays. The concentration of salt in the collection trays can be determined using measurements of conductivity.

2 The role of the Study Group

The results of the experiments show the salt from a single spray to be spread over 5 collections trays, suggesting that the salt collected downstream of the target tray will be recycled through the same spray twice, while those collected upstream will skip one or two sprays and miss some of the bed. Figure 1 shows a typical experimental trace.

The study group was asked to consider these results and determine how to interpret the conductivity traces with regard to the pattern of flow though the cane.

Thus the group was asked to consider the two following questions:

- **MISG:** Can we interpret tracer data from “dumped” salt water input?

- **MISG:** Can we produce a management-generic-interpreter in a spreadsheet?

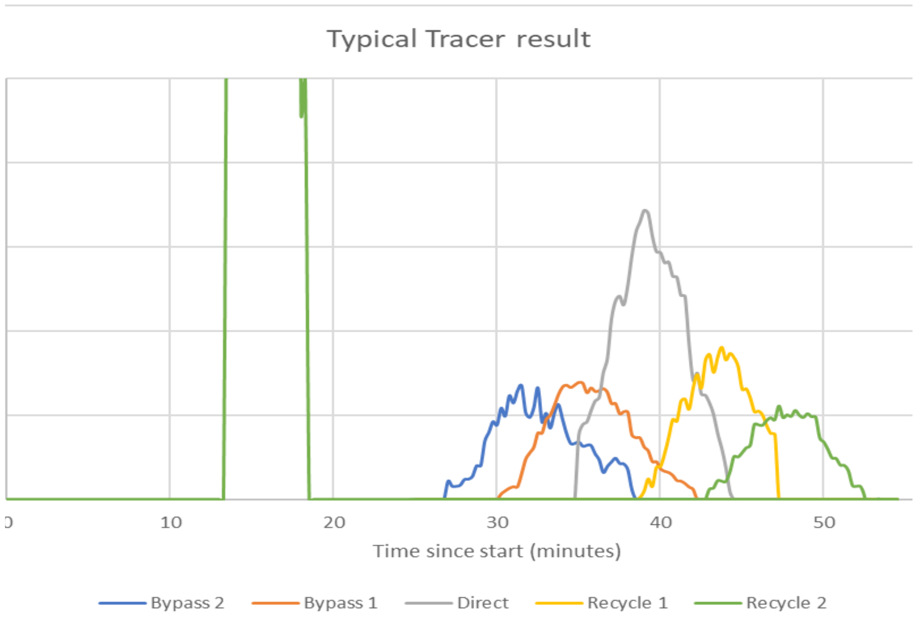

Figure 1: Conductivity from each of the bins subsequent to the injection of saline solution. Bypass bins are upstream and recycle bins are downstream of the expected outflow location.

3 Experiments

3.1 Experimental setup

The results shown in Figure 1 were conducted in the following manner. Fifty kilograms of salt was added to 200 litres of water and the resulting saline solution, with a density of around 1.03 $\mathrm{kg/m^3}$, was then added into the pipe before the sprays over a 5 minute period, giving a rate of $Q_S = 0.05\,\mathrm{m^3/min}$.

Conductivity was measured in the bins over the time of expected arrival of the saline at the bottom of the diffuser. Figure 1 shows the trace from those bins, and it is clear that the saline solution is not only appearing in the expected bin, but also in two bins either side. Further data from other experiments was presented at the study group and some attempt was made to clean this up using a moving filter method.

3.2 Comment

The density of this brine inflow is probably higher than the density of the juice, although it is not clear at which section the saline was added, so it is impossible to know the density of the concurrent juice at this section. This means that it is unlikely that the saline is just a tracer, but will interact with the juice in some way due to the density difference. This introduction of extra fluid may also lead to flooding. These possibilities will be discussed below.

4 A model for flow through the diffuser bed

In order to interpret the output from the experiments, we need some kind of model to enable evaluation of different scenarios. Hocking and Mitchell [3] generated a box model that can be used to study flows of this type, but it is rather complicated and a much clearer idea of flow patterns can be obtained using the model below. As a first attempt, we will use the standard model of porous media flow. In other words, we treat the sugar fibres as a porous medium and consider the flow of the juice through this porous medium. The usual model is based on the definition of the *Piezometric Head*

$$\phi = \frac{p}{\rho g} + y \tag{1}$$

where p is the pressure, ρ is juice density and y is elevation. The flow due to the pressure differential is determined by Darcy's Law

$$\mathbf{q} = (u, v) = -\kappa \nabla \phi, \tag{2}$$

where κ is the so-called "permeability" of the cane pulp. This is combined with the conservation of volume law

$$\nabla \cdot \mathbf{q} = 0 \quad \text{which implies that} \quad \nabla \cdot \kappa \nabla \phi = 0 \tag{3}$$

to provide a closed system. If κ (the permeability) is assumed to be constant then we require

$$\nabla^2 \phi = 0 \tag{4}$$

to be satisfied by the piezometric head. The assumption that the permeability is constant is certainly questionable in this problem, but it allows us to obtain a good idea of the flow lines if the variation in κ is not too large. There are boundary conditions on the height of the water within the bagasse, $y = \eta(x, t)$,

$$\phi = \frac{p(x,t)}{\rho g} + \eta, \quad \text{on} \quad y = \eta(x, t) \tag{5}$$

from the pressure condition, and a kinematic condition

$$\eta_t + u\eta_x - v = 0 \quad \text{on} \quad y = \eta(x,t) \tag{6}$$

that says there can be no flow through the surface of the water. The term $p(x,t)$ represents the pressure on the surface of the cane (of depth D) caused by the presence of the spray jets.

As the liquid egresses from the bottom of the sugar cane, the pressure will match the atmospheric pressure and so

$$\phi = 0 \quad \text{on} \quad y = 0 \tag{7}$$

must also be satisfied.

4.1 Simple linear solution

If we assume the pressure due to the sprays is sinusoidal then

$$\frac{p}{\rho g} = A - A\sin\left(\frac{\pi x}{L} - ct\right) \tag{8}$$

where c is speed of the bed, L is the distance between sprays (length of the trays) and A is the amplitude of the pressure due to the sprays.

If we choose the form

$$\phi(x,y) = \int_0^\infty \sinh(ky)\big[a(k,t)\sin(kx) + b(k,t)\cos(kx)\big]\,\mathrm{d}k \tag{9}$$

and linearize about the cane surface located at $y = \mathrm{D}$ then we find

$$\phi(x,\mathrm{D},t) = A - A\sin\left(\frac{\pi x}{L} - ct\right) + \mathrm{D} \tag{10}$$

and $\eta_t = -\kappa\phi_y(x,\mathrm{D},t)$. We note that (7) satisfied. The solution is

$$\phi(x,y,t) = A\left[\frac{\sinh\frac{\pi y}{L}}{\sinh\frac{\pi \mathrm{D}}{L}}\right]\sin\left(ct - \frac{\pi x}{L}\right)\ . \tag{11}$$

The solution for $\eta(x,y,t)$ is

$$\eta(x,y,t) = y - A\frac{\pi\kappa}{L}\left[\frac{\cosh\frac{\pi y}{L}}{\sinh\frac{\pi \mathrm{D}}{L}}\right]\sin\left(ct - \frac{\pi x}{L}\right). \tag{12}$$

The streamlines can then be computed as

$$\psi(x,y,t) = x + A\left[\frac{\cosh\frac{\pi y}{L}}{\sinh\frac{\pi \mathrm{D}}{L}}\right]\cos\left(ct - \frac{\pi x}{L}\right)\ . \tag{13}$$

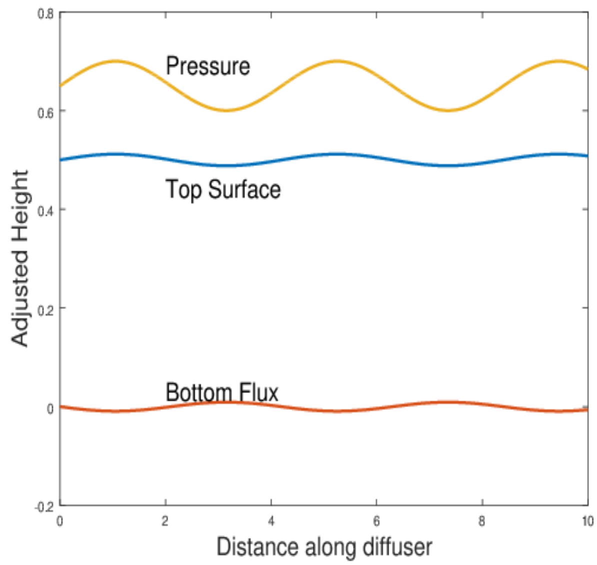

Figure 2: Surface shape and bottom flux for a given pressure (spray distribution) along the length of the diffuser. The level is higher beneath the sprays and then drains away a little before arriving at the next spray jet.

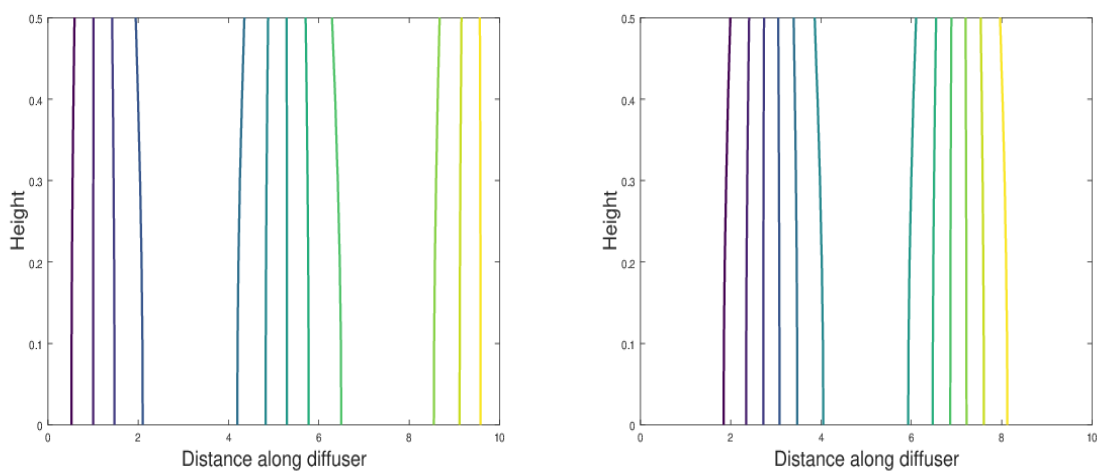

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Figure 3: Streamlines for flow from the sprays through the bed. Closer together means faster flow as the pressure pulse (spray) passes overhead. This view is fixed with the bed, so the sprays are moving to the left.

An example of such a solution is given in Figure 2 and shows the applied “pressure” and the resulting water surface shape and the variation in flux through the base of the bed.

Figure 3 shows the lines of flow and the shape of the surface at two instants in time. Lines close together means higher flow. Note that this solution is in the frame of reference moving with the diffuser, so the sprays are moving from left to right across the top. The lines are close together under the current location of the sprays. Knowing these expected flow lines allows us to interpret the data more clearly.

5 Turbulent diffusion of salt

The high concentration of salt injected into the water during the experiments means that there may be considerable diffusion of the salt as the liquid flows through the bed. A model of the diffusion of salt via turbulent mixing was developed and solved. By tuning the model to estimate the coefficient of diffusion, traces similar to those in the experimental results were found (see Figure 4).

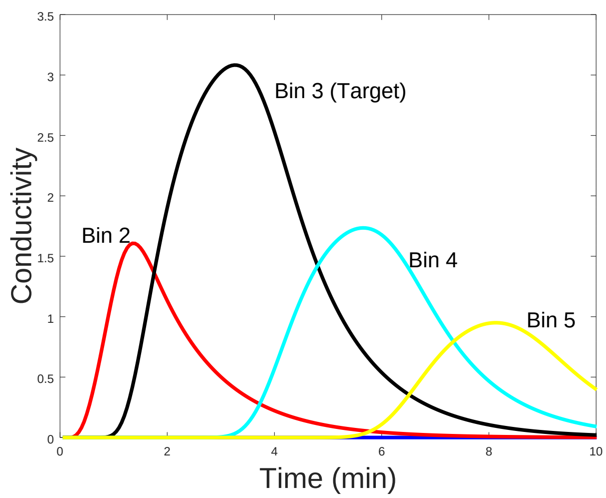

Figure 4: Salt flowing into bins assuming circular disperson as a point source travels downward through the diffuser. Initial source location is in the middle of bin 2, so “aiming” for middle of bin 3.

The estimated value of the diffusion coefficient can be used to interpret the results. In order to produce the traces from the experiments it was found that this rate of diffusion or dispersion of the liquid had to be very large. In fact, from the traces it is not possible to determine whether the salt is spread by mixing of the water or diffusion of the salt or some combination of both. In the view of the group it would be very unlikely that the full spread could be due to water mixing, and that most of it must come from the diffusion of the salt due to turbulent churning of the water (since the coefficient of turbulent diffusion is much higher than that of molecular diffusion of salt). Consider a “blob” of salty water entering at the spray jet. As it travels downward it interacts with other, non-salty water, leading to diffusion of the salt.

In cylindrical polar coordinates the salt diffusion satisfies

$$\frac{\partial S}{\partial t} = \frac{\alpha_S}{r}\frac{\partial^2 S}{\partial r^2} \tag{14}$$

which for a point source gives

$$S(r,t) = \frac{1}{2\alpha_S t}\,\mathrm{e}^{-r^2/4\alpha_S t} \tag{15}$$

for the solution in an unbounded fluid, where α_S is the diffusion rate.

The experiment measured the concentration in the bins, and so for each bin we have the equation

$$\frac{\mathrm{d}S_k}{\mathrm{d}t} = \frac{Q}{V}(F_k(t) - S_k),\ \ k = 1, 2, \ldots, 5, \tag{16}$$

where $F_k(t)$ is the concentration at the time due to the moving source at the top of the respective bins, i.e.

$$F_k(t) = \frac{1}{2\alpha_S t}\int_{X_k}^{X_{k+1}} \mathrm{e}^{-r^2/(4\alpha_S t)}\,\mathrm{d}x. \tag{17}$$

Here X_k is the location of the front of bin k, and

$$r = \sqrt{\big(x - (x_0 - ct)\big)^2 + (\mathrm{D} - ct)^2}$$

measures the distance from the point $(x, 0)$ to the current central point of the source, assuming it began at (x_0, D). The quantity Q/V is the ratio of flux into the bin and the bin volume. These equations were solved with different values of α_S and x_0 to determine the value of the concentration of salt in the bins if the salt spread as though from a single point source.

In Figure 4 the injection point is in the middle of bin 2 and the flow should egress around the middle of bin 3. It is clear that none of the salt reaches (backward) to bin 1, and some reaches each downstream bin with diminishing concentrations.

If, however, the injection location is not exactly correct for the travel speed, then a picture such as Figure 5 emerges. Clearly some of the salt has spread backward to show a small signal at bin 1, much more than expected appears in bin 2 and then it diminishes in subsequent bins as before. It is important to emphasise that if more realistic diffusion values are used the spread would be much less and the salt would appear only in the bins

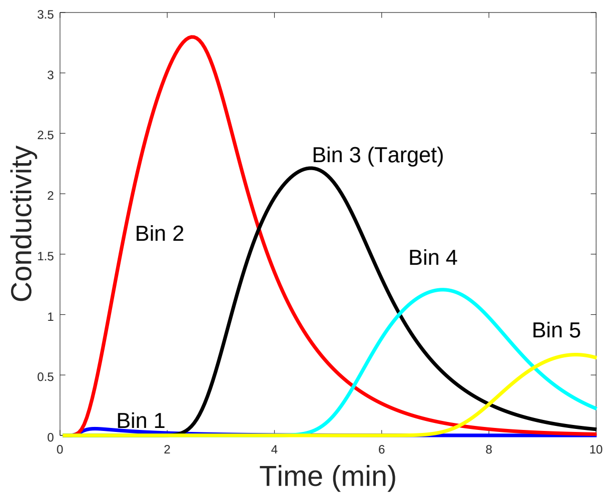

Figure 5: Salt flowing into bins assuming circular disperson as a point source travels downward through the diffuser. Initial source location is at the beginning of bin 2, so should end up flowing out near the end of bin 2 and the start of bin 3.

nearest the "correct" location. This is unlikely to change even allowing for the fact that the injection occured over several minutes.

In both of these figures, $\alpha_S \approx 10^{-3}\ m^2/s$ which is far too large to be realistic. (Note that for molecular diffusion of salt, $\alpha_S \approx 10^{-9}$.) Even allowing for some diffusion due to turbulence this is still far too high.

We do note that the time over which this process occurs is sufficient for juice that has entered one of the bins to reappear upstream. This occurence would no doubt confound the results to a certain extent, since it is not clear how this affects the data traces, and it is not possible to tell the origin of the salt in a particular case. To begin to understand this process, we considered a model for the recycling of a tracer through several spray jets and bins.

5.1 What should non-diffusive tracers look like through several cycles?

Using a standard model of flow in a porous medium, we can compute the location at which a non-reactive tracer would end up in the system. This provides a basis for comparison with the data. The computation assumes that the permeability is constant and the flow through the bed takes approximately the same time along the length of the diffuser.

Given these assumptions there are then three possible outcomes (assuming spread is not too large). All of the tracer can flow out into the correct bin projected, it can "miss" downstream or it can miss "upstream". Here we can model these and see how the tracer profile would look over time.

5.2 Model 1 - "Ideal flow match"

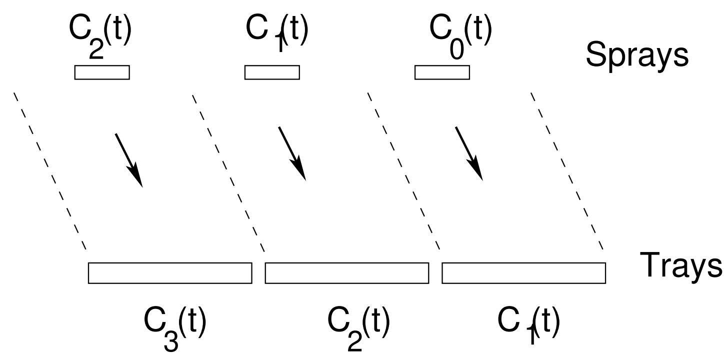

Figure 6: Sketch of an ideal flow case in which all of the tracer enters the "correct" bin.

We consider three bins in a sequence with all of the tracer from input $C_0(t)$ and attempt to track it as it passes through three collection bins. The water from each spray jet enters the corresponding bin for that jet, the water (juice) is collected and fed into the next spray upstream. Let $C_k(t), k = 1, 2, 3, ...,$ be the concentration in each tray, and $C_0(t)$ be the upstream input concentration, then

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = \beta\left(C_0\left(t - \frac{\kappa}{\mathrm{D}}\right) - C_1(t)\right) , \tag{18}$$

$$\frac{\mathrm{d}C_2}{\mathrm{d}t} = \beta\left(C_1\left(t - \frac{\kappa}{\mathrm{D}}\right) - C_2(t)\right) , \tag{19}$$

$$\frac{\mathrm{d}C_3}{\mathrm{d}t} = \beta\left(C_2\left(t - \frac{\kappa}{\mathrm{D}}\right) - C_3(t)\right) , \tag{20}$$

where $\beta = \frac{q}{V}$, κ is the permeability, q the mass flux, V the tray volume and D the fibre depth. The terms containing the function arguments $(t - \kappa/\mathrm{D})$ on the right hand side represent delays of time as the water travels through the bed. It is assumed that there is no time delay between the juice entering the tray and being injected into the next. These delay equations can be solved using Laplace transforms.

The water from each spray jet travels through and enters in exactly the correct downstream bin. In that case, the original signal will be pumped upstream and appear at a later time in the next bin with some dilution.

If the flow is perfectly matched the solution will show a succession of single bumps separated by the time of travel. This can be seen in Figure 7 where the 3 peaks follow exactly from each other.

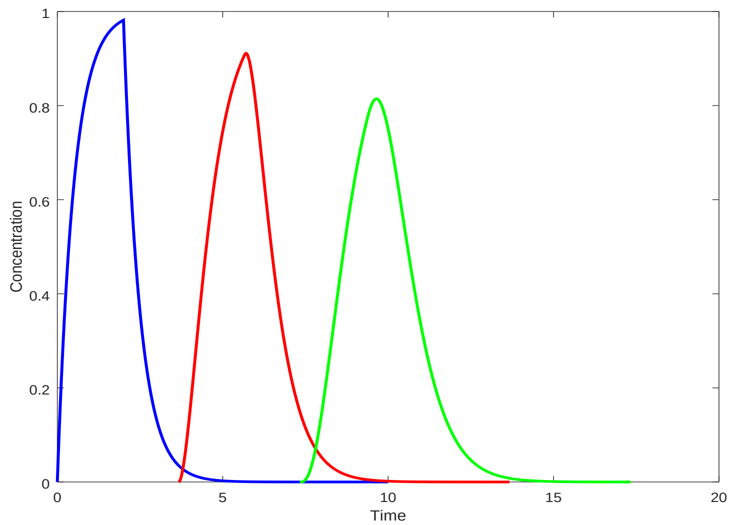

Figure 7: Solutions for bin loads as a function of time for the ideal flow case. The peaks for each bin only occur once as the juice flows directly through and then on to the next bin.

5.3 Model 2 - "Recycle"

In this situation the water from each spray jet enters the corresponding bin for that jet, but some gets into the same bin within which it started. A sketch is shown in Figure 8. The water (juice) is collected and fed into the next spray upstream. The inputs to each tray are now modified as

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = \beta\left((1-\gamma)C_0\left(t-\frac{\kappa}{\mathrm{D}}\right)+\gamma\, C_1\left(t-\frac{\kappa}{\mathrm{D}}\right)-C_1(t)\right)\ , \tag{21}$$

$$\frac{\mathrm{d}C_2}{\mathrm{d}t} = \beta\left((1-\gamma)C_1\left(t-\frac{\kappa}{\mathrm{D}}\right)+\gamma\, C_2\left(t-\frac{\kappa}{\mathrm{D}}\right)-C_2(t)\right)\ , \tag{22}$$

$$\frac{\mathrm{d}C_3}{\mathrm{d}t} = \beta\left((1-\gamma)C_2\left(t-\frac{\kappa}{\mathrm{D}}\right)+\gamma\, C_3\left(t-\frac{\kappa}{\mathrm{D}}\right)-C_3(t)\right)\ , \tag{23}$$

where γ is the amount of overlap of the upstream spray with the "correct" bin, which again can be solved using Laplace transforms or a simple ODE solver. If there is recycling, a second, smaller bump in Bin 1 will appear simultaneously to the primary bump

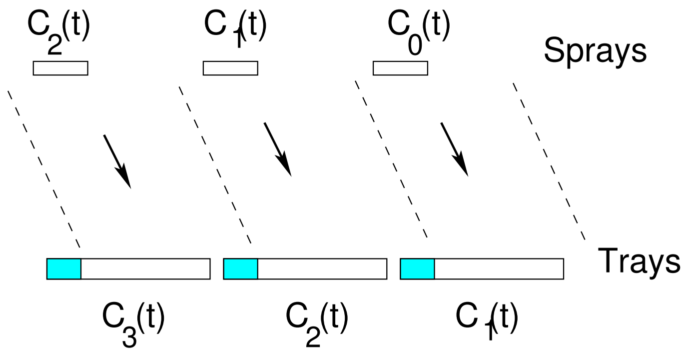

Figure 8: Sketch of case where water enters the downstream bin and hence passes twice through the same spray-bin pair. The shaded portion is that which will be recycled through this bin.

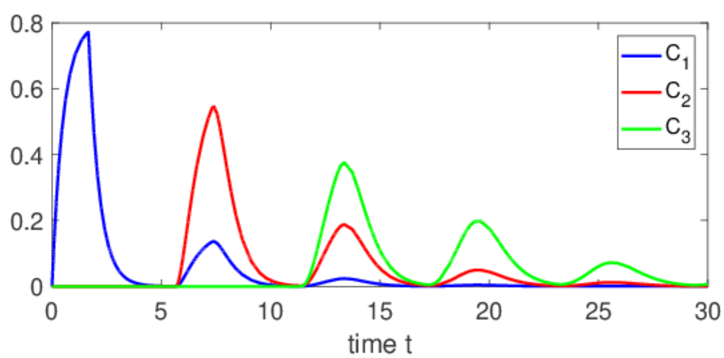

Figure 9: Solutions for bin loads as a function of a time for the recycled flow case. Juice from bin 1 appears at each later time as it recycles through the same bin. Likewise for bin 2.

in Bin 2. This is clearly visible in Figure 9, which shows such a solution when the amount of bin overlap is $\gamma = 0.2$.

5.4 Model 3 - "Bypass"

In the opposite system to the recycle case, the water from each spray jet enters the corresponding bin for that jet, but some gets into the more upstream bin and so bypasses a spray. The water (juice) is collected and fed into the next spray upstream. The inputs

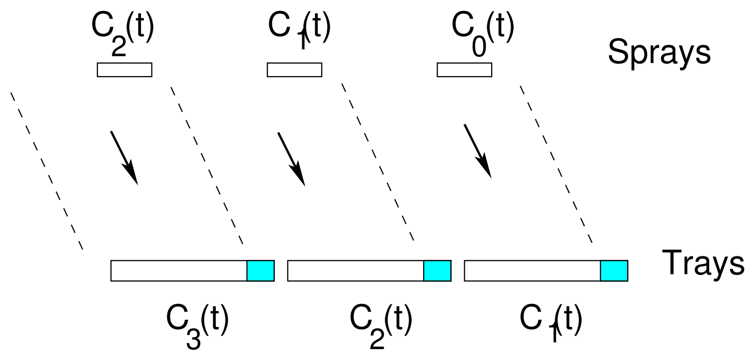

Figure 10: Sketch of case where water enters the upstream bin and hence misses a next spray-bin pair. The shaded part is that which is in the wrong bin.

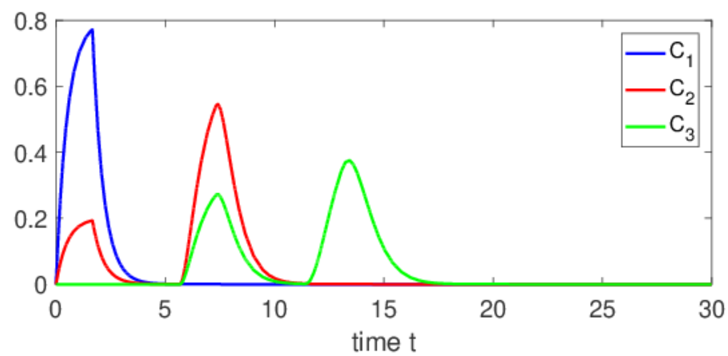

Figure 11: Solutions for bin loads as a function of a time for the bypass flow case. The curve from bin 2 shows a small flow that has bypassed the current bin and consequently appears at an earlier time with the main flow from bin 1. Similarly for bin 3.

to each tray are now modified to be

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = \beta\left((1-\gamma)C_0\left(t-\frac{\kappa}{\mathrm{D}}\right) - C_1(t)\right) \ , \tag{24}$$

$$\frac{\mathrm{d}C_2}{\mathrm{d}t} = \beta\left((1-\gamma)C_1\left(t-\frac{\kappa}{\mathrm{D}}\right) + \gamma\, C_0\left(t-\frac{\kappa}{\mathrm{D}}\right) - C_2(t)\right) \ , \tag{25}$$

$$\frac{\mathrm{d}C_3}{\mathrm{d}t} = \beta\left((1-\gamma)C_2\left(t-\frac{\kappa}{\mathrm{D}}\right) + \gamma\, C_1\left(t-\frac{\kappa}{\mathrm{D}}\right) - C_3(t)\right) \ , \tag{26}$$

where γ is again the overlap, but in the opposite direction. These equations can be solved in the same way. If there is bypass, a first primary bump will appear at Bin 2 simultaneously to the primary from Bin 1.

In general it would seem unlikely that both bypass and recycling would occur at the same time if the system were running close to optimal, but if there is extra spreading of the plume from the spray jet (as identified in the previous section), then it is quite likely that both could occur and secondary peaks would be seen both upstream and downstream of the primary.

6 Concluding remarks

The group has developed several models to try to understand the experimental traces.

The diffuser bed was considered to be an example of a porous medium and the equations for flow through such a medium were implemented. A linearized version of these equations was used to model the flow of the liquid through the diffuser. The model included an approximate determination of the shape of the water surface (the so-called *free surface* or *phreatic surface*) within the cane bed. A sinusoidal pressure distribution across the top of the cane served to represent the spray jets and permitted a series solution to the linear equations.

Secondly, a point source of salt was modelled as it tracked down through the bed, spreading as it went downward due to some form of diffusion. This model was able to generate traces that looked similar to those obtained in the experiments. However, in order to do this a much larger coefficient of diffusion was necessary than would seem realistic. This makes these solutions debatable in spite of their similarity to the aquired traces. As potential further work the point source solution (15) could be used as a Green's function to compute such a spreading flow over a longer time and a wider region such as if there were flooding across the surface of the bed.

A further model was developed to consider the behaviour of a completely benign tracer added to the diffuser and followed for several cycles through the spray/bed/tray system. A set of delay-differential equations resulted and these were solved numerically (and also using Laplace Transforms) to obtain traces of concentration as the tracer travelled through several cycles in the diffuser. This model would enable interpretation of any such future experiments.

There were concerns within the group about the high concentration of salt and the apparently very large diffusion values obtained. High salt concentration may lead to flows other than the flow of the juice due to density differences, perhaps resulting in gravity currents. There were also concerns that the injection of extra water into a diffuser that was running at close to a stable, steady configuration may lead to some flooding at the top of the bed at the point of injection. Water on top of the bed will spread much more quickly than that inside and could therefore distort the results of the experiments quite significantly.

Another possible explanation for the rapid spreading of the salt is that there may be significant unsaturated regions in the diffuser bed, and that this may cause much greater spreading. This possibility would require significant analysis beyond the scope of the study group.

The study group was able to produce simulations that reproduce traces looking similar to the traces shown in the experiments. Given the uncertainty in the physical processes that lead to the traces obtained, however, it was not possible to implement an analysis tool in a spreadsheet during the study group. The interpretation of the results is still an open question and there is more than one possibility in terms of the physics. Turbulent diffusion, a high rate of mixing, flooding, density driven gravity currents and unsaturated flow regions are all possible explanations for the results.

References

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