MODELLING THE SURFACE WATER POLLUTION IN LAKE KIVU

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Abstract

Hazards are posed by the levels of pollutants in the surface layers of Lake Kivu caused primarily by activities around the lake. Some of these activities have their source in the water itself, include municipal sewage and wastewater treatment plants, toxic runoff from farms and factories either near the lake or upstream. Other sources are atmospheric and include gas vented into the atmosphere during harvest time and smokestacks from factories near the lake. The released gases and particles eventually end up in the lake either directly or after rain and can effect the health of inhabitants, animals and water life. The project's aim was to model surface water pollution in the lake.

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1   introduction

Lake Kivu, on the border of Rwanda and the Democratic Republic of Congo, is known as a killer lake primarily because of the vast quantities of carbon dioxide and methane dissolved in the deep waters of the lake which, if spontaneously released, could cause massive damage and loss of life, see the 2018 MISG report[1]. However there are other hazards posed by the levels of pollutants in the surface layers of the lake caused primarily by activities around the lake. Some, having their source in the water itself, include municipal sewage and other effluent entering the lake, wastewater treatment plants, and toxic runoff from farms and factories either near the lake or upstream. Other sources of pollution are atmospheric and include gas vented into the atmosphere during harvest time and smokestacks from factories near the lake. Of particular concern is a cement factory near the shore. The released gases and particles may eventually end up in the lake either directly or after rain and can effect the health of inhabitants, animals and water life. The project's aim was to model surface water pollution in the lake.

The group felt no direct progress could be made on this very worthy problem without specifics. We would need to know what pollutant was of concern together with its possible source, and also data and information concerning the primary factors affecting its dispersal (winds, topography, climate,...). With the above in mind the (small) group decided to set up a generic model that may be used to describe the atmospheric dispersal from a factory chimney. The simple models developed here would then need to be be combined with absorption and dispersal models to determine concentration levels at significant locations in and around the lake. Such models can provide crude estimates for the effect of a known source of pollutant on the environment and are an appropriate start for more substantial investigations. If estimates obtained using such models suggest there is a major concern then more detailed monitoring/computational models would need to be developed. Such models are expensive to develop and maintain, mainly because the data input needed is expensive to collect, and require major and continuous computational monitoring and processing. Such assessments are often required by regulation.

In Section 2 we present well known classical fundamental solution ideas in a chimney release context and then we go on to show how useful solutions can be developed using these ideas. We also indicate the limitations of such an approach. The conclusions are summarised in Section 3.

2 The Fundamental Solution Approach

Depending on specific concerns different approaches and associated models should be set up. The release of pollutant could be a one off event, continuous, intermittent or periodic. The concern could be with pollutant levels soon after release in the immediate neighbourhood of the chimney, or with the long term effects of pollutant release on the environment in the general area. There may be a steady wind, but variability would normally be a feature. The atmosphere in the area may be stable or unstable, with or without an inversion layer at a prescribed height, and all this is likely to change with season. The released gases could either rise and be carried out of the the zone of interest or sink to ground level quickly depending on the temperature and height of release, atmospheric conditions, local geography, time of year.

Also depending on circumstances different fundamental solutions are appropriate. A 1D model would be appropriate for describing the pollutant spread close to the source. A 3D source solution would be appropriate if the region of interest is large compared with the source size (likely) but small compared with the 'atmospheric depth'. A 2D cylindrical source solution would be appropriate if the region of interest is large compared with the effective depth of the atmosphere. Such fundamental solutions may be used to generate the solution for pollutant concentration levels due to quite general source distributions under a range of boundary conditions, and in the presence of wind. The usefulness of the results will depend on the correct choice of model and the detail required. Here we will use a cylindrical model and illustrate how this solution may be used to generate solutions for a broad range of circumstances. We will also indicate the limitations of the procedure.

2.1 The Cylindrical Source Solution

A pollutant is released from a source at time $t = 0$. We will assume the pollutant 'quickly fills' the atmosphere up to an inversion layer of height H and then spreads out cylindrically. In this context the cylindrical source solution is appropriate.

If M is the total mass (kgs) released into the region from a location $x = y = z = 0$ 'instantaneously' (at $t = 0$) then the concentration $c(r, t)$
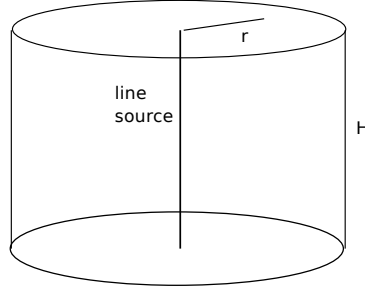
Figure 1: The fundamental cylindrical source solution: Pollutant is released at $r = 0, 0 < z < H$ at $t = 0$ and then spreads out radially. The pollutant concentration is assumed to be uniform in depth through the atmosphere.

(kg/m^3) may be described by the diffusion equation

$$\frac{\partial c}{\partial t} = \kappa \left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} \right) , \text{ with } r^2 = x^2 + y^2, \qquad (2.1)$$

where κ models the turbulent diffusivity of the atmosphere and would need to be determined using experimental data. The total mass, M, is released at $t = 0$, and it is assumed there is no removal which requires

$$H \int_0^\infty c(r, t)(2\pi r dr) = M . \qquad (2.2)$$

This statement of physics can be varified mathematically from the diffusion equation (2.1) and boundary conditions, see Appendix A. No pollutant is initially present in the atmosphere, there is no source at infinity, and the solution is finite at $r = 0$ for $t > 0$, so that

$$c(r,) = 0 , \qquad 0 < r \leq \infty , \qquad (2.3)$$

$$2\pi r \left. \frac{\partial c}{\partial r} (r, t) \right|_{r=0} = 0 , \qquad 2\pi r \left. \frac{\partial c}{\partial r} (r, t) \right|_{r=\infty} = 0 , \qquad t > 0 . \qquad (2.4)$$

Dimensionality arguments then lead to the similarity solution [2]

$$c(r,t) = \frac{M}{4\pi\kappa H t}\, e^{-r^2/(4\kappa t)}, \quad t > 0. \tag{2.5}$$

It is useful to scale the results using:

$$r = r_0 r', \; t = t_0 t', \; \text{where } t_0 = r_0^2/(4\kappa), \tag{2.6}$$

and

$$c = c_0 c'(r', t'), \; \text{with } c_0 = \frac{M}{4\pi\kappa H t_0} \equiv \frac{M}{\pi r_0^2 H}; \tag{2.7}$$

r_0 is the typical distance of interest in context (perhaps 1 km); t_0 then gives the time scale required for pollutant to spread out to this distance from the source with concentration levels of the order of c_0 expected. With this scaling the concentration solution takes the 'universal form'

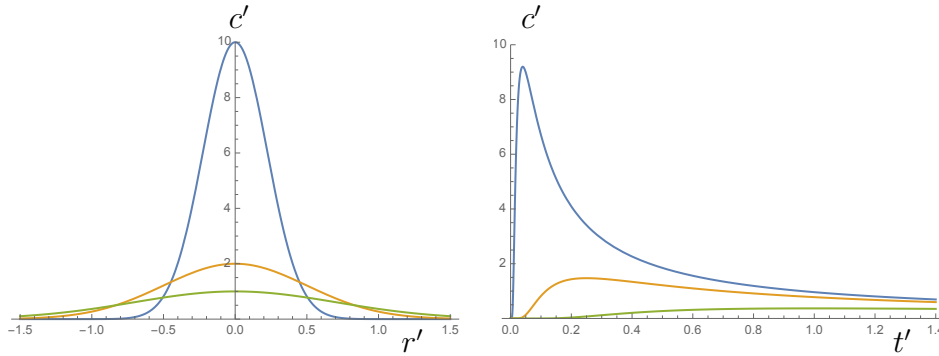$$c'(r', t') = \frac{e^{-r'^2/t'}}{t'}, \tag{2.8}$$

Figure 2: Fundamental source solution: *Left*: Scaled concentration levels as a function of (scaled) distance from the source at scaled times $t' = 0.1$ (top curve) and $t' = 0.5, 1$ (lower curves). *Right*: Concentration levels as a function of time at various distances from the source $r' = 0.2$ (top curve), $r' = 0.5$ and $r' = 1$.

plotted as a function of (scaled) distance r' for various (scaled) times, see Figure 2 *Left*; evidently pollution concentration levels reach a significant

level at r_0 after a time scale of order t_0 as suggested earlier. Also plotted (Figure 2) are concentration levels at fixed locations r' as a function of time t'. Note that concentration level at a specific location $r' = r'_0$ remain 'at zero[2]' until the pollutant reaches that location and then increases, reaching a maximum value of $c'_{max} = e^{-1}/r'^2$ after time $t' = r'^2$ from release.
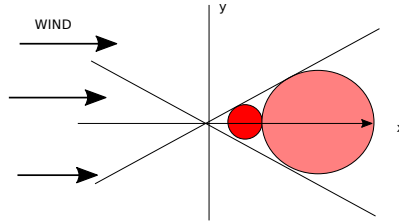
2.2   Wind effects

Figure 3: The effect of a steady wind along the x' axis. The pollutant spreads out in a wedge shaped region.

The above result can now be used to generate solutions of more practical interest. In most circumstance there will be winds blowing and this will dramatically effect the dispersal. If there is a steady speed wind blowing with scaled speed u along the x axis, see Figure 3, the (scaled) concentration profile again due to an instantaneous is given by

$$c'(x', y', t') = \frac{e^{-\left[(x'-ut')^2 + y'^2\right]}}{t'} \; , \tag{2.9}$$

where the scales (r_0, t_0) are chosen as in (2.6). This result is plotted in Figures 4, *Left* and *Right*. The pollutant spreads out in a wedge shaped region centered on the initial source location and spreading out in the wind direction;

The above result assumes a steady and non fluctuating breeze not varying with height above the Earth; boundary layer effects are ignored. Relatively

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[2]Technically speaking the concentration level is non-zero for all $t > 0, r > 0$, however the level is negligibly small for $t' < \epsilon r'$ (with ϵ small); explicitly $c' = \mathcal{O}(e^{-1/\epsilon})$ for $t' < \epsilon r'$.
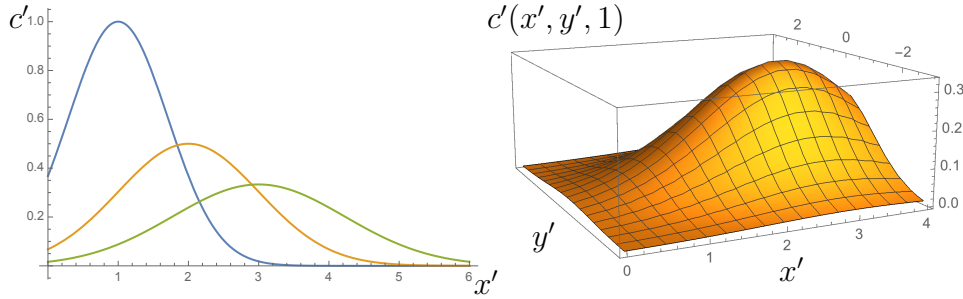
Figure 4: Wind effects: *Left*: Scaled concentration levels $c'(x', y', t')$ along the x' axis in the presence of a (scaled) wind speed $u = 1$ along the x' axis at times $t' = 1$ (left curve), and $t' = 2, 3$. *Right*: Concentration levels $c'(x', y', t')$ at time $t' = 1$.

small scale fluctuations or changes over a long time may be taken into account by suitably modelling the diffusivity but larger scale variations would require a detailed computational/statistical model.

The above result also assumes there is no deposition of pollutant into the ground/water or loss into the upper atmosphere. Such losses may be modelled; a deposition rate proportional to the local concentration level is often assumed. Such a model results in an integral equation for c', which needs to be solved numerically, a relatively simple matter. The resulting model would however need significant tuning to account for near Earth changes in concentration levels.

2.3 Other modifications

If the release rate from the chimney is $q(t)$ kgs/sec then superposition gives the (scaled) result

$$c'(r, t) = \int_0^t q(t') e^{-r^2/(t-t')}/(t - t')dt', \text{ where } r^2 = (x' - ut')^2 + y'^2. \quad (2.10)$$

Of particular interest are situations in which there is a release of pollutant over a prescribed time interval or a periodic release over a day (say). Superposition can also be used to account for the release from distributed sources in space and time.

It is usually the case that there are length scales of different order involved with pollutant dispersal. For example the geographical length scales are usually much larger than the length scales associated with 'diffusion'. Under such circumstances asymptotic techniques may be used to 'separate out', and thus account for, different features of the problem. Typically this will result in multi-scale approximations involving fundamental solutions. Various time scales also arise. For example the pollution diffusive time scale will normally be much shorter than the pollution release time scale or the time scales associated with flow field. Again asymptotic procedures may be useful. The use of such procedures is 'opportunistic' and strongly dependent on the specific circumstances. The advantage of semi-analytic results obtained using such procedures is that they lead to a much better understanding of the effect of various actions on the outcome. For example the maximum pollution concentration levels reached at various locations will normally be greatly effected by the timing the release. A periodic release may give more time for the pollutant to disperse, or separate out, and this may allow for safer release. Of course the release of pollution during the evenings is often used to reduce human exposure.

2.4   Numerical modelling

All such analytic procedures are of limited use, and tuned computational and/or statistical models are the usual next stage. The underlying physics of dispersal is complicated, with the flow field and dispersal processes linked and not well understood (particle movement, buoyancy effects, flow instabilities, statistical fluctuations). A 'complete' Navier-Stokes model might seem to be the way to go but no such complete theory exists and the approximations that are used rely on good judgement. Also such models are computationally intense and numerical error accumulation means that the believable computational time scale is often short compared with the time scale of practical interest. This means that such models are normally just used to examine local features of the dispersal process, for example the flow near the source. More usefully the dimensionless groups associated with the circumstance of interest are identified and experimental/empirical models are used to describe the solution behaviour locally, and then global numerical modals are used to stitch it all together. Of course such models require supporting data and in any case there is no deterministic solution so some sort of statistical modelling is needed.

3   Conclusions

As indicated this very worthy problem requires more specific information to be adequately addressed, and what has been done here just illustrates an approach that may be used to obtain crude estimates for the severity of the pollution problem. More sophisticated computational and statistical models with extensive data input would be required to obtain the results normally needed if pollutant levels are a significant concern.

References

[1] Fowkes N D, Mason D P and Hutchinson A J. Proceedings of the Mathematics in Industry Study Group, South Africa 2018, pp 27-73.

[2] Carslaw H S and Jeager J C. Conduction of Heat in Solids, Oxford at the Clarendon Press, 1959, pp 353-369.

Appendix A

Mathematical background

(i) Conserved quantity

Multiply by r the difusion equation

$$\frac{\partial c}{\partial t} = \frac{\kappa}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c}{\partial r} \right) \tag{A.1}$$

and then integrate it with respect to r from $r = 0$ to $r = \infty$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty r\, c(r,t)\, \mathrm{d}r = \kappa \left[r \frac{\partial c}{\partial r} \right]_0^\infty . \tag{A.2}$$

Impose the boundary conditions

$$r \frac{\partial c}{\partial r} (r,t) \Big|_{r=0} = 0, \qquad r \frac{\partial c}{\partial r} (r,t) \Big|_{r=\infty} = 0 . \tag{A.3}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty r\, c(r,t)\, \mathrm{d}t = 0 \tag{A.4}$$

and therefore

$$\int_0^\infty r\, c(r,t)\, \mathrm{d}r = \text{constant independent of } t. \tag{A.5}$$

In the manuscript the constant is specified and (A.5) is written as

$$2\pi H \int_0^\infty r\, c(r,t)\, \mathrm{d}r = M . \tag{A.6}$$

(ii) Similarity solution of the diffusion equation

The diffusion equation (A.1) is invariant under the scaling transformation

$$\bar{r} = \lambda^a r , \qquad \bar{t} = \lambda^b t , \qquad \bar{c} = \lambda^n c , \tag{A.7}$$

provided $b = 2a$. Suppose that $c = f(r,t)$ is a solution of (A.1). Then $\bar{c} = f(\bar{r}, \bar{t})$ is a solution of

$$\frac{\partial \bar{c}}{\partial \bar{r}} = \frac{\kappa}{\bar{r}} \left(\bar{r} \frac{\partial \bar{c}}{\partial \bar{r}} \right) . \tag{A.8}$$

Thus

$$\lambda^n f(r,t) = f\left(\lambda^a r, \ \lambda^{2a} t\right) \tag{A.9}$$

and differentiating (A.9) by λ and setting $\lambda = 1$ we obtain

$$a\,r\,\frac{\partial f}{\partial r} + 2a\,t\,\frac{\partial f}{\partial t} = n f \ . \tag{A.10}$$

The differential equations of characteristic curves are

$$\frac{\mathrm{d}r}{ar} = \frac{\mathrm{d}t}{2at} = \frac{\mathrm{d}f}{nf} \ . \tag{A.11}$$

Two independent solutions are

$$\frac{r}{t^{1/2}} = k_1 \ , \qquad \frac{f}{t^{n^*}} = k_2 \ , \tag{A.12}$$

where $n^* = n/a$ and k_1 and k_2 are constants. The general solution of (A.10) is $k_2 = F(k_1)$ where F is an arbitrary function. Since $c = f(r,t)$ it follows that

$$c(r,t) = t^{n^*} F(\xi) \ , \qquad \xi = \frac{r}{t^{1/2}} \ . \tag{A.13}$$

Substituting (A.13) into (A.1) reduces the diffusion equation to the ordinary differential equation

$$\frac{\kappa}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi}\right) + \frac{1}{2}\,\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi} - n^* F = 0 \ . \tag{A.14}$$

In order to obtain n^* we substitute (A.13) into the conserved quantity (A.6) which becomes

$$2\pi\,H\,t^{n^*+1} \int_0^\infty \xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi}\,\mathrm{d}\xi = M \ . \tag{A.15}$$

Since H and M are constants it follows that $n^* = -1$ and

$$c(r,t) = \frac{1}{t}\,F(\xi) \ , \qquad \xi = \frac{r}{t^{1/2}} \ . \tag{A.16}$$

The differential equation (A.14) becomes

$$2\kappa\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi}\right) + \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 F\right) = 0 \ . \tag{A.17}$$

This illustrates the double reduction theorem that if a partial differential equation is reduced to an ordinary differential equation by a scaling transformation associated with a conserved quantity then the ordinary differential equation can be integrated at least once. Integrating (A.17) gives

$$2\kappa\,\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi} + \xi^2\,F = A \,, \tag{A.18}$$

where A is a constant. But from the first boundary condition in (A.3),

$$\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi}\Big|_{\xi=0} = 0 \tag{A.19}$$

and therefore $A = 0$. Thus

$$2\kappa\,\xi\,\frac{\mathrm{d}F}{\mathrm{d}\xi} + \xi^2\,F = 0 \tag{A.20}$$

which is a variables separable first order differential equation with solution

$$F(\xi) = B\,\exp\left(-\frac{\xi^2}{4\kappa}\right) \,. \tag{A.21}$$

The constant B is obtained from the conserved quantity. By substituting (A.21) into (A.15) with $n^* = -1$ we obtain

$$2\pi\,HB\int_0^\infty \xi\exp\left(-\frac{\xi^2}{4\kappa}\right)\,\mathrm{d}\xi = M \tag{A.22}$$

and integrating gives

$$B = \frac{M}{4\pi\kappa H} \,. \tag{A.23}$$

Hence

$$c(r,t) = \frac{M}{4\pi\,\kappa H}\,\frac{1}{t}\,\exp\left(-\frac{r^2}{4\kappa t}\right) \,, \tag{A.24}$$

in agreement with (2.1) in the manuscript.