Modelling Rogue Waves
Wake of Destruction

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Types of oceanic waves

- Tsunamis - generated by earthquakes
- Surface-gravity waves - Wind-generated
- Rogue waves - ?
Rogue waves - also known as “freak”, “monster” or “abnormal” waves - are waves whose amplitude is unusually large for a given sea state.

- Unexpected and known to appear and disappear suddenly.
- Also occur in optical fibers, atmospheres and plasmas.
Size Comparison

**Figure:** The size comparison between a large rogue wave, a seven storey building, a giraffe and an average human being.
The Draupner wave, New Year's Day 1995. Using a laser, the Draupner oil platform in the North Sea measured a wave with height of 25.6m.

In February 2000, an oceanographic research vessel recorded a wave of height 29m in Scotland.

3-4 large oil tankers are badly damaged yearly when traveling the Agulhas current off the coast of South Africa.
Causes of rogue waves

- Wave-wave interaction
- Wave-current interaction
- Spatial focusing
- Focusing due to nonlinearity
Recap - Linear Causing Mechanisms

- Geometrical or Spatial Focusing
- Wave-Current Interaction
- Focusing due to Dispersion
Solution

\[
\Psi = -\frac{Hg \cosh(k(h+z))}{2\sigma} \frac{\sin(kx - \sigma t)}{\cosh(kh)}
\]

\[
\eta = \frac{H}{2} \cos(kx - \sigma t) \text{ at } z = 0
\]

\[
\sigma^2 gk \tanh(kh)
\]
Characteristics

- Gaussian bell shaped
- Higher amplitude than normal
- Travels long distances without breaking
- Breaks inside ocean
Solitary Waves

- **Solitary Waves** are solutions to these equations, occurring when there is a balance of the *dispersive* and *nonlinear* effects.

- We are dealing with the Nonlinear Shrödinger Equation, which is considered a **Non Linear Evolution Equation.**
Consider the Korteweg-de Vries (KdV) equation,

\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} = 0, \]  

(1)

where \( \frac{\partial^3 u}{\partial x^3} \) is the dispersive term, which causes the wave to "spread out", whilst the nonlinear term, \( 6u \frac{\partial u}{\partial x} \) effectively causes the wave to resist this effect. This balance creates a solitary wave.
What is a Soliton?

A **soliton** is a solitary wave which maintains its shape when it moves at a constant speed and conserves amplitude, shape, and velocity after a collision with another soliton.

Solitons differ from breathers, which are waves that oscillate in time (breathe). Kinks represent waves with a steep inclination. All these are also solutions to nonlinear wave equations.
Soliton Types

- In deep water conditions there are three accepted solutions to the Nonlinear Schrödinger Equation in the form of solitons. Solitons are the accepted rogue wave models.

- The deep water condition for a rogue wave is that $kh > 1.36$ where $k$ is the wavenumber and $h$ is the water depth.
Assumptions and Approximations

- Constant density $\rho$ - fair assumption
- Wavelength $\lambda >$ amplitude $A$ - fair until the wave is rogue
- Negligible viscosity - fair for ocean water
- Irrotational flow - perhaps an approximation
- Only body force is gravity - fair
- The water is deep and the bed flat - fair
Homogenous Nonlinear Schrödinger Equation

\[ \frac{i}{2} \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial \xi^2} + k |\psi|^2 \psi = 0 \]

\( t \): Time

\( \xi \): Distance
Coupled System - Manakov System

\[ i \frac{\partial \psi}{\partial \tau} + \frac{\partial^2 \psi}{\partial x^2} + 2k(|\psi|^2 + |\phi|^2)\psi = 0 \]

\[ i \frac{\partial \phi}{\partial \tau} + \frac{\partial^2 \phi}{\partial x^2} + 2k(|\psi|^2 + |\phi|^2)\phi = 0 \]
Coupled non-linear Shrödinger equation

Figure: Collision of two waves at angle $\theta$
Coupled non-linear Shrödinger equation

\[
\frac{\partial A}{\partial t} = - C_x \frac{\partial A}{\partial x} - C_y \frac{\partial A}{\partial y} + i \left( \alpha \frac{\partial^2 A}{\partial x^2} + \beta \frac{\partial^2 A}{\partial y^2} + \gamma \frac{\partial A}{\partial x \partial y} \right) - i (\zeta |A|^2 A + 2\zeta |B|^2 A)
\]

\[
\frac{\partial B}{\partial t} = - C_x \frac{\partial B}{\partial x} - C_y \frac{\partial B}{\partial y} + i \left( \alpha \frac{\partial^2 B}{\partial x^2} + \beta \frac{\partial^2 B}{\partial y^2} + \gamma \frac{\partial B}{\partial x \partial y} \right) - i (\zeta |B|^2 B + 2\zeta |A|^2 B)
\]
Collision of Waves

The angular frequency,

\[ \omega = \sqrt{gk} , \]

and the magnitude of the wave vector,

\[ \kappa = \sqrt{k_x^2 + k_y^2} . \]

where,

\[ k_x = \kappa \cos (\theta) , \]
Collision of Waves

and,

\[ k_y = \kappa \sin(\theta). \]

\[ \zeta = \frac{\omega (k_x^5 - k_x^3 k_y^2 - 3 k_x k_y^4 - 2 k_x^4 \kappa + 2 k_x^2 k_y^2 \kappa + 2 k_y^4 \kappa)}{2 \kappa^2 (k_x - 2 \kappa)} \]
Collision of Waves

Once the waves collide, the angular frequency becomes,

\[ \Omega = \pm \sqrt{\tau K^2 \left[ (\xi (A_0^2 + B_0^2) + \tau K^2) \pm \sqrt{\xi^2 (A_0^2 + B_0^2)^2 + 16\zeta^2 A_0^2 B_0^2} \right]} \]
Solution To Nonlinear Schrödinger Equation

\[ \psi(\xi, t) = \left[ 1 + \frac{2(1 - 2a) \cosh(b\xi) + ib\sinh(b\xi)}{\sqrt{2a} \cos(wt) - \cosh(b\xi)} \right] e^{i\xi}, \]

where

\[ b = \sqrt{8a(1 - 2a)} \quad \quad \quad w = 2\sqrt{1 - 2a} \]
Waves Types by Varying $a$

$0 < a < 0.5$ – Akhmediev Breather

$a \rightarrow 0.5$ – Peregrine Soliton

$0.5 < a < \infty$ – Kuznetsov-Ma Soliton
An Upgrade: The Dysthe Model

Still assumes negligible viscosity and incompressibility, solutions are still solitons and breathers that go to plane waves at $\pm \infty$

\[
\frac{\partial \phi_1}{\partial t} + \frac{1}{2} \frac{\partial \phi_1}{\partial x} + \frac{i}{8} \frac{\partial^2 \phi_1}{\partial x^2} - \frac{i}{4} \frac{\partial^2 \phi_1}{\partial y^2} - \frac{1}{16} \frac{\partial^3 \phi_1}{\partial x^3} + \frac{3}{8} \frac{\partial^3 \phi_1}{\partial x \partial y^2} - \frac{5i}{128} \frac{\partial^4 \phi_1}{\partial x^4} - \frac{15i}{32} \frac{\partial^4 \phi_1}{\partial x^2 \partial y^2} - \frac{3i}{32} \frac{\partial^4 \phi_1}{\partial y^4} + \frac{i}{2} |\phi_1|^2 \phi_1 + \frac{7}{256} \frac{\partial^5 \phi_1}{\partial x^5} - \frac{35}{64} \frac{\partial^5 \phi_1}{\partial x^3 \partial y^2} + \frac{21}{64} \frac{\partial^5 \phi_1}{\partial x \partial y^4} + \frac{3}{2} |\phi_1|^2 \frac{\partial \phi_1}{\partial x} - \frac{1}{4} \phi_1^2 \frac{\partial \phi_1^*}{\partial x} + i\phi_1 \frac{\partial \phi_0}{\partial x} = 0
\]
The KdV equation for a soliton in shallow water in one dimension,

\[
\frac{\partial v}{\partial t} - \frac{3}{2} \sqrt{\frac{g}{h}} v \frac{\partial v}{\partial x} - \frac{h^2}{6} \sqrt{gh} \frac{\partial^3 v}{\partial x^3} = 0
\]  

(2)

Using the following relations to remove the dimensions,

\[
u = u, \quad t \rightarrow \frac{1}{6} \sqrt{\frac{g}{h}} t, \quad x \rightarrow \frac{x}{h} - t
\]
After making the variables dimensionless, the standard form of the KdV is,

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3}
\]

The solution for a soliton is,

\[
u(x, t) = -\frac{c}{2} \text{sech}^2 \left[ \frac{\sqrt{c}}{2} (x - ct - x_0) \right] \tag{4}\]
Figure: The plot of the solution to the one dimensional KdV equation
Two dimensional, nonlinear, Korteweg-de Vries

\[
\frac{\partial}{\partial x_1} \left( \frac{\partial u}{\partial t} + h \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial F(u)}{\partial x_1} \right) = f \frac{\partial^2 u}{\partial x_2^2} \tag{5}
\]

Where the stream function is represented by,

\[
F(u) = au + \frac{b}{2} u^2 - \frac{1}{3} u^3 \tag{6}
\]
The solution is then

\[
    u(t, x_1, x_2) = \frac{b}{2d} \pm \frac{k_1 \sqrt{6h}}{\sqrt{d}} \tanh \left( \frac{t(-4adk_1^2 - b^2 k_1^2 + 4dfk_2^2 + 8dhk_1^4)}{4dk_1} + k_1 x_1 + k_2 x_2 + k_3 \right)
\]  

(7)
Two dimensional, nonlinear, Korteweg-de Vries

The following boundary conditions are imposed on equation 7

\[ u_{0,0,0} = 0, \quad \frac{\partial^3 u}{\partial x_2^3}(0,0,0) = 0, \quad \frac{\partial^3 u}{\partial x_2^3}(0,0,0) = 0 \]

\[ t > 0 \quad L_1 > 0 \quad L_2 > 0 \]

\[ x_1 \in (0, L_1) \quad x_2 \in (0, L_2) \]

where \( u \) has the parameters \( u(t, x_1, x_2) \).
Two dimensional, nonlinear, Korteweg-de Vries

This allows us to obtain 3 equations to solve for $k_1, k_2, k_3$

\[ b + 2\sqrt{6dh}k_1 \tanh(k_3) = 0 \]

\[ 4k_1^3 \tanh^2(k_3) \text{sech}^2(k_3) - 2k_1^3 \text{sech}^4(k_3) = 0 \]

\[ 4k_2^3 \tanh^2(k_3) \text{sech}^2(k_3) - 2k_2^3 \text{sech}^4(k_3) = 0 \]
The 2D, nonlinear KdV equation

Figure: The plot of the solution to the two dimensional KdV equation
The 2D, nonlinear Kadomtsev-Petviashvili equation

Particular case of the KdV equation, with a term $\gamma$ which is dependent on the dispersion medium

$$\frac{\partial}{\partial x_1} \left( \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x_1^3} - 6u \frac{\partial u}{\partial x_1} \right) = 3\gamma^2 \frac{\partial^2 u}{\partial x_2^2}$$  \hspace{1cm} (8)$$

The solution is a hyperbolic function,

$$u(t, x_1, x_2) = \alpha + \beta \tanh^2(k_1 x_1 + k_2 x_2 + k_0 + t \omega)$$  \hspace{1cm} (9)$$
Where,

\[ \alpha = -\frac{(3\gamma^2 k_2^2 - k_1 \omega - 8k_1^4)}{6k_1^2}, \quad \beta = -2k_1^2 \] (10)

The \( k \) constants can then be determined as above, with the boundary conditions
The 2D, nonlinear Kadomtsev-Petviashvili equation

Figure: The plot of the solution to the two dimensional KP equation
Animations

and

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