Modelling Rogue Waves

Wake of Destruction

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Types of oceanic waves

- Tsunamis - generated by earthquakes
- Surface-gravity waves - Wind-generated
- Rogue waves - ?

Rogue waves

- Rogue waves - also known as “freak”, “monster” or “abnormal” waves - are waves whose amplitude is unusually large for a given sea state.
- Unexpected and known to appear and disappear suddenly.
- Also occur in optical fibers, atmospheres and plasmas.

Size Comparison

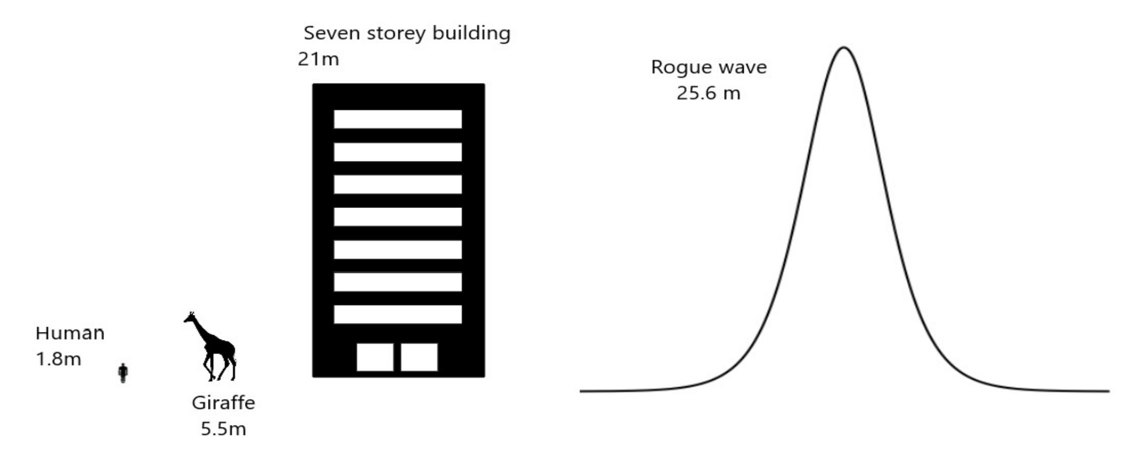


Figure: The size comparison between a large rogue wave, a seven storey building, a giraffe and an average human being.

- The Draupner wave, New Year's Day 1995. Using a laser, the Draupner oil platform in the North Sea measured a wave with height of 25.6m
- In February 2000, an oceanographic research vessel recorded a wave of height 29m in Scotland
- 3-4 large oil tankers are badly damaged yearly when traveling the Agulhas current off the coast of South Africa.

Causes of rogue waves

- Wave-wave interaction
- Wave-current interaction
- Spatial focusing
- Focusing due to nonlinearity

Recap - Linear Causing Mechanisms

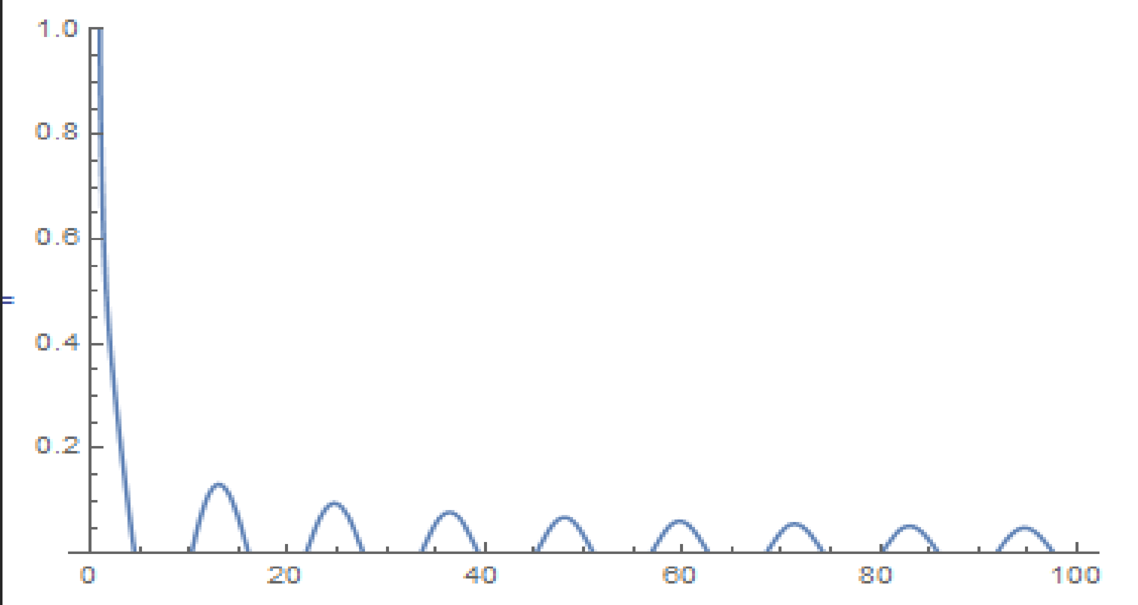
- Geometrical or Spatial Focusing
- Wave-Current Interaction
- Focusing due to Dispersion

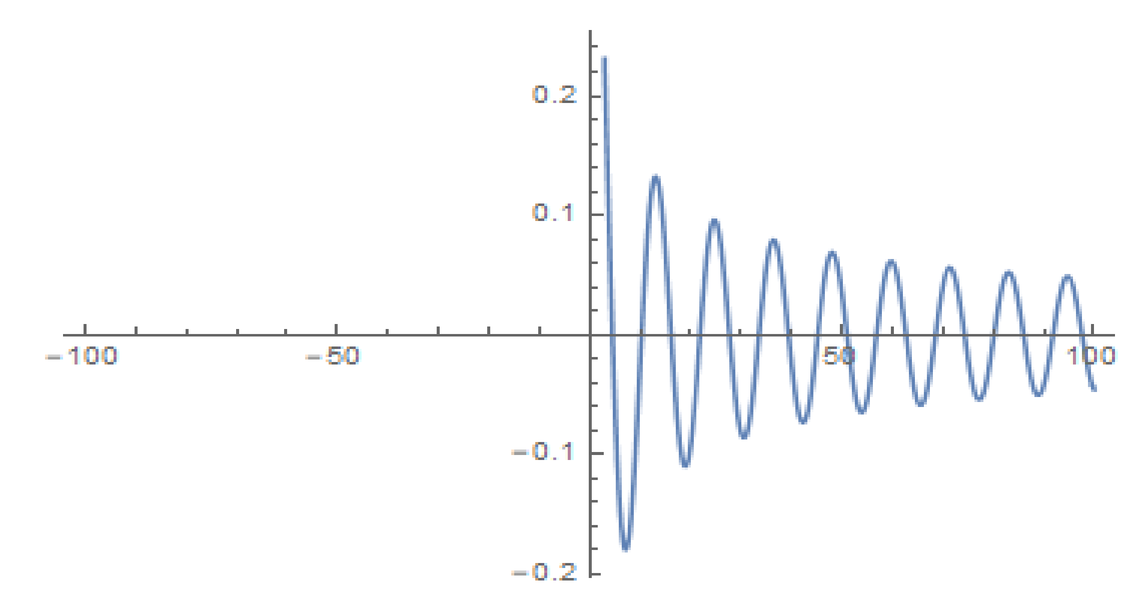
Solution

$$\Psi = -\frac{Hg}{2\sigma}\frac{\cosh(k(h+z))}{\cosh(kh)}\sin(kx - \sigma t)$$

$$\eta = \frac{H}{2}\cos(kx - \sigma t) \text{ at } z = 0$$

$$\sigma^2 gk\tanh(kh)$$


1.0
0.8
0.6
0.4
0.2
0
20
40
60
80
100


0.2
0.1
−0.1
−0.2
−100
−50
50
100

Characteristics

- Gaussian bell shaped
- Higher amplitude than normal
- Travels long distances without breaking
- Breaks inside ocean

Solitary Waves

- **Solitary Waves** are solutions to these equations, occurring when there is a balance of the *dispersive* and *nonlinear* effects.
- We are dealing with the Nonlinear Shrödinger Equation, which is considered a **Non Linear Evolution Equation**.

Korteweg-de Vries (KdV) equation

Consider the Korteweg-de Vries (KdV) equation,

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0, \tag{1}$$

where $\frac{\partial^3 u}{\partial x^3}$ is the dispersive term, which causes the wave to "spread out", whilst the nonlinear term, $6u\frac{\partial u}{\partial x}$ effectively causes the wave to resist this effect. This balance creates a solitary wave.

What is a Soliton?

A **soliton** is a solitary wave which maintains its shape when it moves at a constant speed and conserves amplitude, shape, and velocity after a collision with another soliton.

Solitons differ from breathers, which are waves that oscillate in time (breathe). Kinks represent waves with a steep inclination. All these are also solutions to nonlinear wave equations.

Soliton Types

- In deep water conditions there are three accepted solutions to the Nonlinear Schrödinger Equation in the form of solitons. Solitons are the accepted rogue wave models.
- The deep water condition for a rogue wave is that $kh > 1.36$ where k is the wavenumber and h is the water depth.

Assumptions and Approximations

- Constant density ρ - fair assumption
- Wavelength λ > amplitude A - fair until the wave is rogue
- Negligible viscosity - fair for ocean water
- Irrotational flow - perhaps an approximation
- Only body force is gravity - fair
- The water is deep and the bed flat -fair

Homogenous Nonlinear Schrödinger Equation

$$\frac{i}{2}\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial \xi^2} + k\left|\psi\right|^2 \psi = 0$$

t : Time

ξ : Distance

Coupled System - Manakov System

$$i\frac{\partial \psi}{\partial \tau} + \frac{\partial^2 \psi}{\partial x^2} + 2k(|\psi|^2 + |\phi|^2)\psi = 0$$

$$i\frac{\partial \phi}{\partial \tau} + \frac{\partial^2 \phi}{\partial x^2} + 2k(|\psi|^2 + |\phi|^2)\phi = 0$$

Coupled non-linear Shrödinger equation

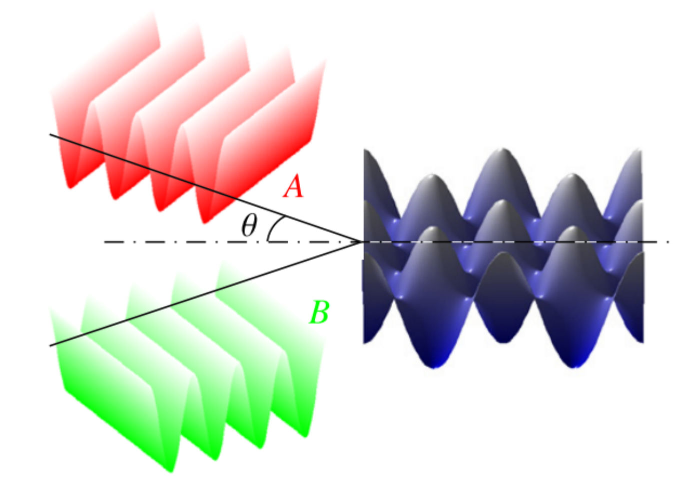


Figure: Collision of two waves at angel θ

Coupled non-linear Shrödinger equation

$$\frac{\partial A}{\partial t} = -C_x\frac{\partial A}{\partial x} - C_y\frac{\partial A}{\partial y} + i\left(\alpha\frac{\partial^2 A}{\partial x^2} + \beta\frac{\partial^2 A}{\partial y^2} + \gamma\frac{\partial A}{\partial x\partial y}\right) - i\left(\zeta|A|^2 A + 2\zeta|B|^2 A\right)$$

$$\frac{\partial B}{\partial t} = -C_x\frac{\partial B}{\partial x} - C_y\frac{\partial B}{\partial y} + i\left(\alpha\frac{\partial^2 B}{\partial x^2} + \beta\frac{\partial^2 B}{\partial y^2} + \gamma\frac{\partial B}{\partial x\partial y}\right) - i\left(\zeta|B|^2 B + 2\zeta|A|^2 B\right)$$

Collision of Waves

The angular frequency,

$$\omega = \sqrt{gk},$$

and the magnitude of the wave vector,

$$\kappa = \sqrt{k_x^2 + k_y^2}.$$

where,

$$k_x = \kappa \cos(\theta),$$

Collision of Waves

and,

$$k_y = \kappa \sin(\theta)\,.$$

$$\zeta = \frac{\omega(k_x^5 - k_x^3 k_y^2 - 3k_x k_y^4 - 2k_x^4\kappa + 2k_x^2 k_y^2\kappa + 2k_y^4\kappa)}{2\kappa^2(k_x - 2\kappa)}$$

Collision of Waves

Once the waves collide, the angular frequency becomes,

$$\Omega = \pm\sqrt{\tau K^2 \left[\left(\xi \left(A_0^2 + B_0^2\right) + \tau K^2\right) \pm \sqrt{\xi^2 \left(A_0^2 + B_0^2\right)^2 + 16\zeta^2 A_0^2 B_0^2} \right]}$$

Solution To Nonlinear Schrödinger Equation

$$\psi(\xi, t) = \left[1 + \frac{2(1-2a)\cosh(b\xi) + \mathrm{i}b\sinh(b\xi)}{\sqrt{2a}\cos(wt) - \cosh(b\xi)}\right] e^{\mathrm{i}\xi},$$

where

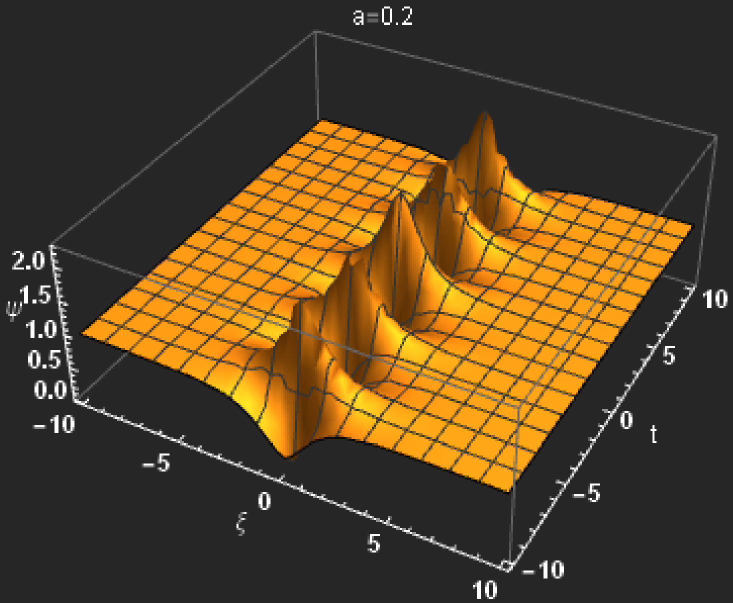
$$b = \sqrt{8a(1-2a)} \qquad w = 2\sqrt{1-2a}$$

Waves Types by Varying a

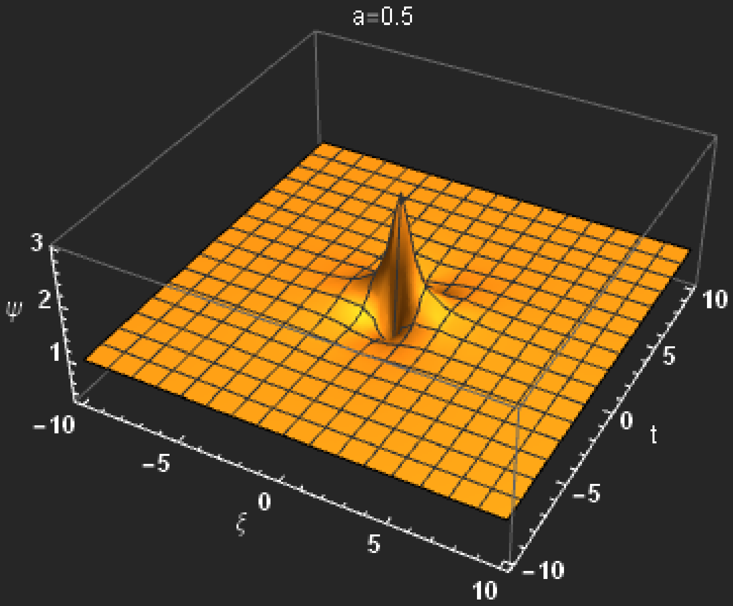
$0 < a < 0.5$ – **Akhmediev Breather**

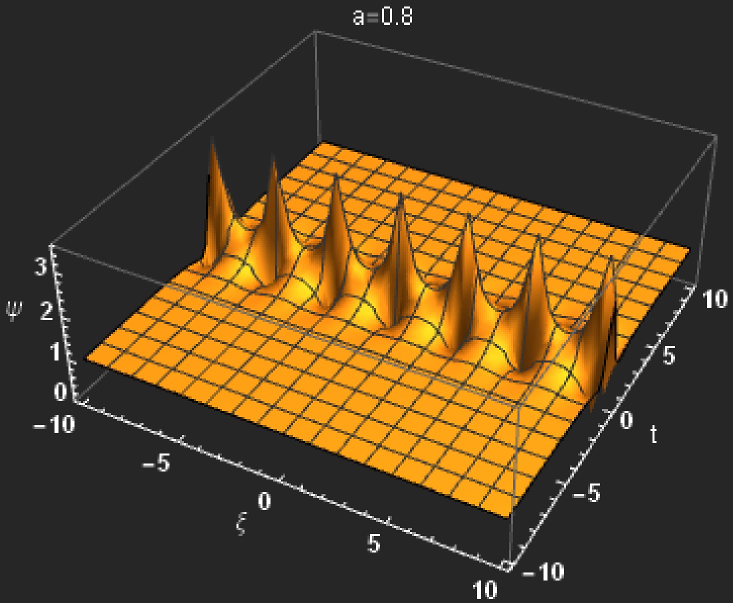
$a \to 0.5$ – **Peregrine Soliton**

$0.5 < a < \infty$ – **Kuznetsov-Ma Soliton**



a=0.2
2.0
1.5
1.0
0.5
0.0
ε
−10
−5
0
5
10
ξ
−10
−5
0
5
10
t


a=0.5
3
2
ψ
1
−10
−5
0
ξ
5
10
−10
−5
0
t
5
10


a=0.8
3
2
1
0
ψ
−10
−5
0
5
10
ξ
−10
−5
0
5
10
t

An Upgrade: The Dysthe Model

- Still assumes negligible viscosity and incompressibility, solutions are still solitons and breathers that go to plane waves at $\pm\infty$

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\frac{\partial \phi_1}{\partial x} + \frac{i}{8}\frac{\partial^2 \phi_1}{\partial x^2} - \frac{i}{4}\frac{\partial^2 \phi_1}{\partial y^2} - \frac{1}{16}\frac{\partial^3 \phi_1}{\partial x^3} + \frac{3}{8}\frac{\partial^3 \phi_1}{\partial x \partial y^2} - \frac{5i}{128}\frac{\partial^4 \phi_1}{\partial x^4}$$

$$-\frac{15i}{32}\frac{\partial^4 \phi_1}{\partial x^2 \partial y^2} - \frac{3i}{32}\frac{\partial^4 \phi_1}{\partial y^4} + \frac{i}{2}|\phi_1|^2\phi_1 + \frac{7}{256}\frac{\partial^5 \phi_1}{\partial x^5} - \frac{35}{64}\frac{\partial^5 \phi_1}{\partial x^3 \partial y^2} + \frac{21}{64}\frac{\partial^5 \phi_1}{\partial x \partial y^4}$$

$$+\frac{3}{2}|\phi_1|^2\frac{\partial \phi_1}{\partial x} - \frac{1}{4}\phi_1^2\frac{\partial \phi_1^*}{\partial x} + i\phi_1\frac{\partial \phi_0}{\partial x} = 0$$

One dimensional, nonlinear, Korteweg-de Vries

The KdV equation for a soliton in shallow water in one dimension,

$$\frac{\partial v}{\partial t} - \frac{3}{2}\sqrt{\frac{g}{h}}\, v\frac{\partial v}{\partial x} - \frac{h^2}{6}\sqrt{gh}\,\frac{\partial^3 v}{\partial x^3} = 0 \tag{2}$$

Using the following relations to remove the dimensions,

$$u = \frac{v}{h} \qquad t \rightarrow \frac{1}{6}\sqrt{\frac{g}{h}}\, t \qquad x \rightarrow \frac{x}{h} - t$$

One dimensional, nonlinear, Korteweg-de Vries

After making the variables dimensionless, the standard form of the KdV is,

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3}$$

The solution for a soliton is,

$$u(x,t) = -\frac{c}{2} sech^2 \left[\frac{\sqrt{c}}{2}(x - ct - x_0) \right] \tag{4}$$

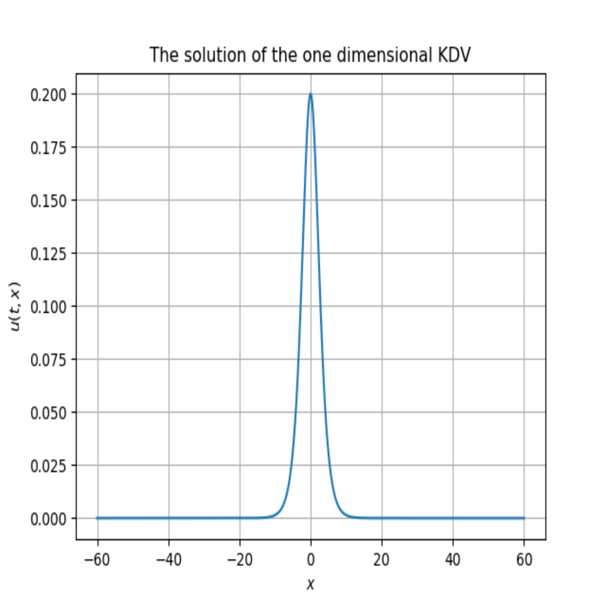


Figure: The plot of the solution to the one dimensional KdV equation

Two dimensional, nonlinear, Korteweg-de Vries

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial t}+h\frac{\partial^3 u}{\partial x_1^3}+\frac{\partial F(u)}{\partial x_1}\right)=f\frac{\partial^2 u}{\partial x_2^2} \tag{5}$$

Where the stream function is represented by,

$$F(u)=au+\frac{b}{2}u^2-\frac{1}{3}du^3 \tag{6}$$

Two dimensional, nonlinear, Korteweg-de Vries

The solution is then

$$u(t,x_1,x_2) = \frac{b}{2d} \pm \frac{k_1\sqrt{6h}}{\sqrt{d}} \tanh\left(\frac{t(-4adk_1^2 - b^2k_1^2 + 4dfk_2^2 + 8dhk_1^4)}{4dk_1} + k_1x_1 + k_2x_2 + k_3\right) \quad (7)$$

Two dimensional, nonlinear, Korteweg-de Vries

The following boundary conditions are imposed on equation 7

$$u_{0,0,0} = 0, \quad \frac{\partial^3 u}{\partial x_2^2}|_{(0,0,0)} = 0, \quad \frac{\partial^3 u}{\partial x_2^2}|_{(0,0,0)} = 0$$

$$t > 0 \qquad L_1 > 0 \qquad L_2 > 0$$

$$x_1 \in (0, L_1) \qquad x_2 \in (0, L_2)$$

where u has the parameters $u(t, x_1, x_2)$.

Two dimensional, nonlinear, Korteweg-de Vries

This allows us to obtain 3 equations to solve for $k_1,\ k_2,\ k_3$

$$b + 2\sqrt{6dh}\,k_1 \tanh(k_3) = 0$$

$$4k_1^3 \tanh^2(k_3)\,sech^2(k_3) - 2k_1^3\,sech^4(k_3) = 0$$

$$4k_2^3 \tanh^2(k_3)\,sech^2(k_3) - 2k_2^3\,sech^4(k_3) = 0$$

The 2D, nonlinear KdV equation

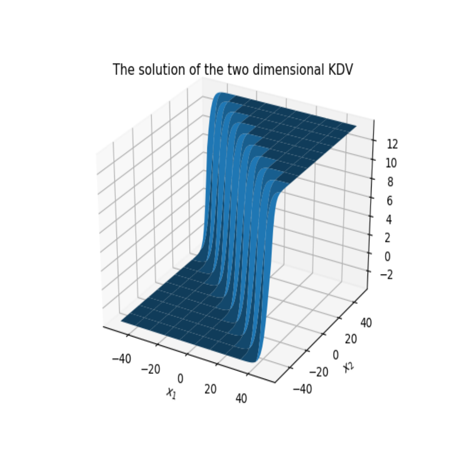


Figure: The plot of the solution to the two dimensional KdV equation

The 2D, nonlinear Kadomtsev-Petviashvili equation

Particular case of the KdV equation, with a term γ which is dependent on the dispersion medium

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x_1^3} - 6u\frac{\partial u}{\partial x_1}\right) = 3\gamma^2 \frac{\partial^2 u}{\partial x_2^2} \tag{8}$$

The solution is a hyperbolic function,

$$u(t, x_1, x_2) = \alpha + \beta \tanh^2(k_1 x_1 + k_2 x_2 + k_0 + t\omega) \tag{9}$$

The 2D, nonlinear Kadomtsev-Petviashvili equation

Where,

$$\alpha = -(3\gamma^2 k_2^2 - k_1\omega - 8k_1^4)/6k_1^2, \qquad \beta = -2k_1^2 \tag{10}$$

The k constants can then be determined as above, with the boundary conditions

The 2D, nonlinear Kadomtsev-Petviashvili equation

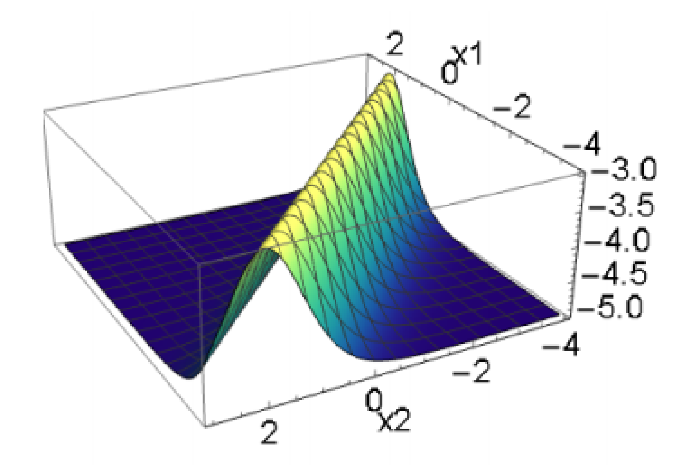


Figure: The plot of the solution to the two dimensional KP equation

Animations

and

Acknowledgements