MODELLING AIR BLASTS IN A LONG TUNNEL WITH SURFACE ROUGHNESS

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Abstract

Collapsing rock due to induced excavations in mining cause air blasts which can propagate for large distances in connecting tunnels. The Fanno model for turbulent flow in a tunnel is applicable if the tunnel is long enough for wall drag to be significant. This may be satisfied in a mining environment where underground excavations are connected by networks of tunnels and shafts. Conservation laws for three partial differential equations derived from the Fanno model are investigated. For each equation the elementary conservation law is obtained. For two of the equations a second conservation law is found. It is demonstrated how conserved quantities derived from the conservation laws and boundary conditions can be used to obtain similarity solutions. Conserved quantities may also be useful when checking the accuracy of numerical solutions.

1 Introduction

During mining operations large masses of rock are induced to fall in excavations or cavities. This collapsing rock causes air blasts which can propagate

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for large distances in connecting tunnels. The air blasts are extremely hazardous because they can overturn vehicles and cause considerable damage to mining infrastructure [1]

In the Second Mathematics in Industry Study Group (2005) models were developed by Sjöberg *et.al.* [2] for the increase in air pressure due to the collapse of rock in an underground excavation. It was found that the rise in the air pressure at the entrance to the tunnel is not very large and it was concluded that the pressure at the mouth of the tunnel is of the form

$$p = p_0\big(1 + \varepsilon\tilde{p}(t)\big) \tag{1.1}$$

where p_0 is the pressure in the cavity before collapse.

The Mathematics in Industry Study Group 2005 did not complete the investigation of the dynamics of the air flow in a connecting tunnel but suggested that the Fanno model for turbulent compressible flow in a rough walled tunnel may be applicable [3]. In the mining environment tunnels are generally sufficiently long for turbulent dissipation to be significant.

The Study Group in 2006 was therefore asked to investigate if the parameters for typical mining procedures lead to the Fanno regime and if this is the case to investigate the application of the Fanno model and asymptotic reductions of the model to the air flow in a tunnel connected to the excavation.

Air blasts have been observed to travel large distances in tunnel networks and this suggests that conservation laws and conserved quantities derived from the conservation laws and boundary conditions may be important in understanding the dynamics of the turbulent air flow in the tunnel. Conservation laws for the partial differential equations of the asymptotic reductions of the Fanno model will therefore be investigated as part of the analysis.

2 Fanno flow model and Fanning friction factor

The essential assumption in the Fanno flow model is that the main effect of turbulence is to exert a wall drag by way of a boundary layer at the wall. The wall drag dampens flow over large times and distances but it has a small effect locally.

The Fanno flow model will be applicable if the flow is turbulent and the tunnel is long enough for wall drag to be important. These conditions were satisfied in the problem of air-jet spinning of polymer filaments for which the two-dimensional channel had aspect ratio 10^{-3} [3, 4]. In the mining situation the diameter of a tunnel may be 4 m and an aspect ratio of 10^{-3} would correspond to a tunnel network of about 4 km in length. In many mining situations there are usually interconnections of an underground excavation by tunnels and shafts to other underground excavations. The Fanno flow model may therefore be applicable.

In order to illustrate the derivation of the equations for Fanno flow in a tube, Ockendon *et.al.* [3] considered the simpler problem of flow in a two-dimensional channel. We will briefly outline here the main steps of their derivation with emphasis on the part played by the wall drag in the Fanno model.

Consider a nearly unidirectional two-dimensional gas flow in a channel $-\frac{1}{2}\,S(x) < y < \frac{1}{2}\,S(x)$ whose length is much greater than its width. Let the mean turbulent velocity, pressure and density be (u^*, v^*), p^* and ρ^*. The mean turbulent stress tensor is

$$\tau^*_{xx} = p^* \,, \quad \tau^*_{yy} = p^* \,, \quad \tau^*_{xy} = \tau^*_{yx} = \tau^* \,. \tag{2.1}$$

The equations of conservation of mass and the balance equations for momentum and energy in a form suitable for averaging over the width of the channel are

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial}{\partial x}(\rho^* u^*) + \frac{\partial}{\partial y}(\rho^* v^*) = 0 \,, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho^* u^*) + \frac{\partial}{\partial x}(\rho^* u^{*2}) + \frac{\partial}{\partial y}(\rho^* u^* v^*) + \frac{\partial p^*}{\partial x} = \frac{\partial \tau^*}{\partial y} \,, \tag{2.3}$$

$$\frac{\partial}{\partial t}(\rho^* E^*) + \frac{\partial}{\partial x}(\rho^* u^* E^*) + \frac{\partial}{\partial y}(\rho^* v^* E^*) + \frac{\partial}{\partial x}(u^* p^*) + \frac{\partial}{\partial y}(v^* p^*) = \frac{\partial}{\partial y}(u^* \tau^*) \,. \tag{2.4}$$

We average over the channel by integrating with respect to y from $y = -\frac{1}{2}\,S(x)$ to $y = \frac{1}{2}\,S(x)$. It is assumed that τ^* changes rapidly from a small value outside the boundary layers to the wall stress τ_W at the walls and that, except for the right hand sides of (2.3) and (2.4), all the terms in the equations can be considered as being independent of y when averaging over the channel. We assume that there is no slip and no blowing at the boundaries $y = \pm\frac{1}{2}\,S$. Thus

$$u^*(x, \pm\tfrac{1}{2}\,S) = 0 \,, \qquad v^*(x, \pm\tfrac{1}{2}\,S) = 0 \,. \tag{2.5}$$

Then for example,

$$\int_{-\frac{1}{2}S}^{\frac{1}{2}S} \frac{\partial u^*}{\partial x}\, dy = \frac{\partial}{\partial x}\left(u^* \int_{-\frac{1}{2}S}^{\frac{1}{2}S} dy \right) = \frac{\partial}{\partial x}(u^* S) \,, \tag{2.6}$$

$$\int_{-\frac{1}{2}S}^{\frac{1}{2}S} \frac{\partial v^*}{\partial y}\, dy = v^*(x, \tfrac{1}{2}S) - v^*(x, -\tfrac{1}{2}S) = 0 \,, \tag{2.7}$$

$$\int_{-\frac{1}{2}S}^{\frac{1}{2}S} \frac{\partial}{\partial y}\,(u^*\tau^*)dy = u^*(x, \tfrac{1}{2}S)\tau^*(x, \tfrac{1}{2}S) - u^*(x, -\tfrac{1}{2}S)\tau^*(x, -\tfrac{1}{2}S) = 0 \,, \tag{2.8}$$

$$\int_{-\frac{1}{2}S}^{\frac{1}{2}S} \frac{\partial}{\partial y}\,\tau^* dy = \tau^*(x, \tfrac{1}{2}S) - \tau^*(x, -\tfrac{1}{2}S) = 2\tau_W \,, \tag{2.9}$$

where

$$\tau^*(x, \tfrac{1}{2}S) = \tau_W, \qquad \tau^*(x, -\tfrac{1}{2}S) = -\tau^*(x, \tfrac{1}{2}S) = -\tau_W \,. \tag{2.10}$$

We denote by u, v, p, ρ, E the average of the quantity over the channel and since u^*, v^*, p^*, ρ^* and E^* are independent of y in the averaging process, $u = u^*$, $v = v*$, $p = p^*$, $\rho = \rho^*$ and $E = E^*$. Thus averaging (2.2) to (2.4) over the channel gives

$$\frac{\partial}{\partial t}\,(\rho S) + \frac{\partial}{\partial x}\,(\rho u S) = 0 \,, \tag{2.11}$$

$$\frac{\partial}{\partial t}\,(\rho u S) + \frac{\partial}{\partial x}\,(\rho u^2 S) + \frac{\partial}{\partial x}\,(p S) = 2\tau_W \,, \tag{2.12}$$

$$\frac{\partial}{\partial t}\,(\rho E S) + \frac{\partial}{\partial x}\,(\rho u E) + \frac{\partial}{\partial x}\,(u p S) = 0 \,. \tag{2.13}$$

The only difference between the Fanno flow model and laminar gas flow is the right hand side of the momentum balance equation, (2.12). The right hand side of the energy balance equation, (2.13), vanishes as in inviscid laminar gas flow because of the no slip boundary condition at the wall. The shear stress at the wall, τ_W, is given by the empirical result

$$\frac{\tau_W}{\frac{1}{2}\,\rho u|u|} = -f \,, \tag{2.14}$$

where $f = \mathrm{O}(10^{-3})$ is a Fanning-type fraction factor.

Ockendon *et.al.* [3] set

$$E = \frac{p}{(\gamma - 1)\rho} + \frac{1}{2}\,u^2 \,, \tag{2.15}$$

where γ is the ratio of the specific heats. The equations are non-dimensionalised using a typical channel width S_0, a channel length $L = S_0/f$ and time L/u_0 where u_0 is a typical initial value of u. The length scale is the minimum channel length over which the effect of wall shear stress enters the model at zero order. There are no Riemann invariants as in inviscid gas flow. The wall drag can cause shocks even in a channel with constant width.

The equations for Fanno gas flow in a tube are similar with S denoting the cross-sectional area.

Two problems considered by Ockendon *et.al.* [3] may be relevant to air blasts. The first problem is concerned with turbulent gas flow in a tube when the pressure at the end $x = 0$ is suddenly changed by an amount small compared with the background pressure. This could model the sudden increase in pressure at the tunnel entrance due to the fall of the rock mass in the excavation. The second is the piston problem in which a piston at $x = 0$ is moved impulsively with constant velocity small compared with the speed of sound in the undisturbed gas. Plugs are placed at the entrance to tunnels connecting with the excavation before the rock mass is induced to fall. If the plug is not put firmly in place it may be moved along the tunnel like a piston by the air blast. The piston problem could model the turbulent gas flow due to the motion of the plug along the tunnel.

3 Small pressure change at tunnel entrance

The gas is initially at rest with pressure p_0 and density ρ_0 in a semi-infinite tunnel $x > 0$. At $t = 0$, the pressure at the entrance is changed from p_0 to $p_0(1 + \varepsilon\gamma)$ where $\varepsilon > 0$. Ockendon *et.al.* [3] introduced the time variable τ and the pressure and density variables, $\tilde{p}$ and $\tilde{\rho}$, defined by

$$t = \varepsilon\tau\,, \qquad p = \tfrac{1}{\varepsilon}\tilde{p}\,, \qquad \rho = 1 + \varepsilon\tilde{\rho}\,. \tag{3.1}$$

A shock moves into the undisturbed region. The equation of the shock in the (x, τ) plane is

$$x = \left(1 + \frac{1}{4}(1 + \gamma)\right)\tau\,. \tag{3.2}$$

Ockendon *et.al.* [3] considered a sequence of time scales. The wall drag has a different effect over each time scale. They found that the two most important time regimes are $\tau = O(\varepsilon^{-1})$, which is the scale over which the wall drag first affects the lowest order solution, and $\tau = O(\varepsilon^{-2})$. We will consider the time scale $\tau = O(\varepsilon^{-2})$. For this time scale the solution is easier to analyse asymptotically.

Consider first the flow near the shock. Introduce the time and length scales, τ_2 and ε_2, defined by

$$\tau_2 = \varepsilon^2\tau\,, \qquad x_2 = \varepsilon^2 x\,, \tag{3.3}$$

which implies that $x = O(\varepsilon^{-2})$, and the velocity u_2, pressure p_2 and density ρ_2 defined by

$$u = \varepsilon u_2 , \qquad \tilde{p} = \varepsilon p_2 , \qquad \tilde{\rho} = \varepsilon \rho_2 . \tag{3.4}$$

Then the second order conservation of mass equation and momentum and energy balance equations in x_2 and τ_2 are

$$\frac{\partial \rho_2}{\partial \tau_2} + \frac{\partial u_2}{\partial x_2} = 0 , \tag{3.5}$$

$$\frac{\partial u_2}{\partial \tau_2} + \frac{\partial p_2}{\partial x_2} = -u_2^2 , \tag{3.6}$$

$$\frac{\partial p_2}{\partial \tau_2} + \frac{\partial u_2}{\partial x_2} = 0 . \tag{3.7}$$

The pressure can be eliminated from (3.6) and (3.7) to give a nonlinear wave equation for u_2:

$$\frac{\partial^2 u_2}{\partial \tau_2^2} - \frac{\partial^2 u_2}{\partial x_2^2} = -2u_2 \frac{\partial u_2}{\partial \tau_2} . \tag{3.8}$$

The boundary conditions on u_2 are

$$x_2 = \tau_2 : \qquad u_2 = \frac{2}{\tau_2} , \tag{3.9}$$

$$x_2 \to 0 : \qquad u_2 \sim \frac{3\tau_2}{x_2^2} . \tag{310}$$

Also at second order, the pressure $p_2 \to \infty$ as $x_2 \to 0$. By considering how p_2 grows as $x_2 \to 0$, Ockendon *et.al.* [3] introduced the variables $\bar{x}_2$ and $\bar{u}_2$ defined by

$$x_2 = \varepsilon^{1/3} \bar{x}_2 , \qquad u_2 = \varepsilon^{-2/3} \bar{u}_2 , \tag{3.11}$$

which implies that $x = O\left(\varepsilon^{-5/3}\right)$ and $u = O\left(\varepsilon^{1/3}\right)$, and transformed from p_2 and ρ_2 back to $\tilde{p}$ and $\tilde{\rho}$ defined by (3.4). Equations (3.5) to (3.7) transform to

$$\frac{\partial \tilde{\rho}}{\partial \tau_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = 0 , \tag{3.12}$$

$$\frac{\partial \tilde{p}}{\partial \bar{x}_2} = -\bar{u}_2^2 , \tag{3.13}$$

$$\frac{\partial \tilde{p}}{\partial \tau_2} + \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = 0 . \tag{3.14}$$

The velocity $\bar{u}_2$ can now be eliminated from (3.13) and (3.14) to give a nonlinear diffusion equation for $\tilde{p}$:

$$\frac{\partial \tilde{p}}{\partial \tau_2} = \frac{1}{2\left(-\frac{\partial \tilde{p}}{\partial \bar{x}_2}\right)^{1/2}} \frac{\partial^2 \tilde{p}}{\partial \bar{x}^2} . \tag{3.15}$$

The pressure $\tilde{p}$ satisfies the boundary conditions

$$\bar{x}_2 = 0 : \qquad \tilde{p} = 1 , \tag{3.16}$$

$$\bar{x}_2 \to \infty : \qquad \tilde{p} \sim \frac{3\tau_2^2}{\bar{x}_2^3} . \tag{3.17}$$

Alternatively. the pressure $\tilde{p}$ can be eliminated from (3.13) and (3.14) to give a nonlinear diffusion equation for $\bar{u}_2$:

$$\frac{\partial^2 \bar{u}_2}{\partial \bar{x}_2^2} = 2\bar{u}_2 \frac{\partial \bar{u}_2}{\partial \tau_2} . \tag{3.18}$$

The velocity $\bar{u}_2$ satisfies the boundary conditions

$$\bar{x}_2 = 0 : \qquad \frac{\partial \bar{u}_2}{\partial \bar{x}_2} = 0 , \tag{3.19}$$

$$\bar{x}_2 \to \infty : \qquad \bar{u}_2 \sim \frac{3\tau_2}{\bar{x}_2^2} . \tag{3.20}$$

The solution of (3.15) to (3.17) for $\tilde{p}$ and (3.18) to (3.20) for $\bar{u}_2$ apply in a region nearer the entrance where $x = \mathrm{O}(x^{-5/3})$. Equation (3.8) for u_2 and bounding conditions (3.9) and (3.10) apply in an adjacent region next to the shock where $x = \mathrm{O}(\varepsilon^{-2})$.

We can therefore summarise the structure of the solution when $\tau = \mathrm{O}(\varepsilon^{-2})$ [3]. When $x = \mathrm{O}(\tau) = \mathrm{O}(\varepsilon^{-2})$, which is the region close to the shock, $\tilde{p} = \mathrm{O}(\varepsilon)$ and $u = \mathrm{O}(\varepsilon)$ (since $\tilde{p} = \varepsilon p_2$ and $u = \varepsilon u_2$ and p_2 and u_2 are O(1) in this region). Equation (3.8) and boundary conditions (3.9) and (3.10) apply. When $x = \mathrm{O}(\varepsilon^{-5/3})$ which is the region further from the shock and nearer the entrance, $\tilde{p}$ increases to O(1) and u to $\mathrm{O}(\varepsilon^{1/3})$ (since $u = \varepsilon^{1/3}\bar{u}_2$ and $\bar{u}_2 = \mathrm{O}(1)$ in this region). Equation (3.15) with boundary conditions (3.16) and (3.17) and equation (3.18) with boundary conditions (3.19) and (3.20) apply.

4 Similarity solutions

The similarity solution of the nonlinear diffusion equation (3.15) for $\tilde{p}$ subject to the boundary conditions (3.16) and (3.17) has been derived by Ockendon *et.al.* [3]. We will briefly outline the steps in the derivation because intermediate results will be required later. We will then show how conservation laws and conserved quantities derived from the conservation laws can be used to solve problem (3.18) to (3.20) for the velocity $\bar{u}_2$ and obtain another solution of (3.15) for $\tilde{p}$. This will motivate the study of conservation laws in the next section. The approach of Dresner [5] for the derivation of similarity solutions will be followed.

To simplify the notation we will write (3.15) to (3.17) as

$$\frac{\partial p}{\partial t} = \frac{1}{2\left(-\frac{\partial p}{\partial x}\right)^{1/2}} \frac{\partial^2 p}{\partial x^2}\,, \tag{4.1}$$

$$x = 0: \qquad p = 1, \tag{4.2}$$

$$x \to \infty: \qquad p \sim \frac{3t^2}{x^3} \tag{4.3}$$

and (3.18) to (3,20) as

$$\frac{\partial^2 u}{\partial x^2} = 2u\,\frac{\partial u}{\partial t}\,, \tag{4.4}$$

$$x = 0: \qquad \frac{\partial u}{\partial x} = 0, \tag{4.5}$$

$$x \to \infty: \quad u \sim \frac{3t}{x^2}. \tag{4.6}$$

4.1 Diffusion equation for pressure

Consider the scaling transformation

$$\bar{t} = \lambda^a t\,, \qquad \bar{x} = \lambda^b x\,, \qquad \bar{p} = \lambda^c p\,. \tag{4.7}$$

Under the transformation (4.7), (4.1) becomes

$$\frac{\partial \bar{p}}{\partial \bar{t}} = \lambda^{-a+\frac{3}{2}b+\frac{1}{2}c} \frac{1}{2\left(-\frac{\partial \bar{p}}{\partial \bar{x}}\right)^{1/2}} \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} \tag{4.8}$$

and therefore (4.1) is invariant under the transformation (4.7) provided $c = 2a - 3b$. Suppose that the solution of (4.1) is $p = f(t, x)$. Then

$$\bar{p} = f(\bar{t}, \bar{x}) \tag{4.9}$$

is the solution of

$$\frac{\partial \bar{p}}{\partial \bar{t}} = \frac{1}{2\left(-\frac{\partial \bar{p}}{\partial \bar{x}}\right)^{1/2}} \frac{\partial^2 \bar{p}}{\partial \bar{x}^2} \tag{4.10}$$

because (4.10) has the same form as (4.1). From (4.9)

$$\lambda^{2a-3b} f(t, x) = f(\lambda^a t, \lambda^b x) \tag{4.11}$$

and differentiating (4.11) with respect to λ and dropping the overhead bars or equivalently setting $\lambda = 1$ gives

$$at \frac{\partial f}{\partial t} + bx \frac{\partial f}{\partial x} = (2a - 3b) f \ . \tag{4.12}$$

The solution of the first order linear partial differential equation for $f(t, x)$ is readily derived and since $p = f(t, x)$ we obtain

$$p(t, x) = t^{2-3\alpha} F(\xi) \ , \qquad \xi = \frac{x}{t^\alpha} \ , \tag{4.13}$$

where F is an arbitrary function and $\alpha = b/a$.

Substituting (4.13) into the partial differential equation (4.1) yields the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + 2\alpha\xi \left(-\frac{dF}{d\xi}\right)^{1/2} \frac{dF}{d\xi} - 2(2-3\alpha)\left(-\frac{dF}{d\xi}\right)^{1/2} F = 0 \ . \qquad 4.14)$$

The constant α is obtained from non-homogeneous boundary conditions or from conserved quantities.

Consider the boundary condition $p = 1$ at $x = 0$ which using (4.13) is

$$1 = t^{2-3\alpha} F(0) \ . \tag{4.15}$$

Thus $\alpha = \frac{2}{3}$ and

$$p(t, x) = F(\xi) \ , \qquad \xi = \frac{x}{t^{2/3}} \ , \tag{4.16}$$

where $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + \frac{4}{3}\xi\left(-\frac{dF}{d\xi}\right)^{1/2} \frac{dF}{d\xi} = 0 \ , \tag{4.17}$$

subject to the boundary conditions

$$F(0) = 1\,, \tag{4.18}$$

$$F(\xi) \sim \frac{3}{\xi^3} \qquad \text{as } \xi \to \infty\,. \tag{4.19}$$

Equation (4.17) is a first order differential equation in $\frac{dF}{d\xi}$. Letting

$$W = -\frac{dF}{d\xi}\,, \tag{4.20}$$

equation (4.17) becomes the separable equation

$$\frac{dW}{d\xi} = -\frac{4}{3}\xi W^{3/2} \tag{4.21}$$

which is readily solved to give

$$\frac{dF}{d\xi} = -\frac{9}{(\xi^2 + a^2)^2}\,, \tag{4.22}$$

where a is a constant. Equation (4.22) may be integrated by letting $\xi = a\tan\theta$ and imposing the boundary condition $F(0) = 1$, we obtain

$$F(\xi) = 1 - \frac{9}{2a^3}\left[\frac{a\xi}{(\xi^2 + a^2)} + \tan^{-1}\left(\frac{\xi}{a}\right)\right]. \tag{4.23}$$

But [6]

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + \mathrm{O}\left(\frac{1}{x^5}\right)\,, \quad \text{as} \quad x \to \infty \tag{4.24}$$

and therefore

$$F(\xi) = 1 - \frac{9\pi}{4a^2} + \frac{3}{\xi^3} + \mathrm{O}\left(\frac{1}{\xi^5}\right)\,, \quad \text{as} \quad \xi \to \infty\,. \tag{4.25}$$

It follows from the boundary condition (4.19) that

$$a = \left(\frac{9\pi}{4}\right)^{1/3} \tag{4.26}$$

and hence

$$p(t,x) = F(\xi) = 1 - \frac{3}{\pi}\left[\frac{\zeta}{1+\zeta^2} + \tan^{-1}\zeta\right]\,, \tag{4.27}$$

where

$$\zeta = \left(\frac{4}{9\pi}\right)^{1/3} \xi = \left(\frac{4}{9\pi}\right)^{1/3} \frac{x}{t^{2/3}} . \tag{4.28}$$

Equation (4.27) is the solution derived by Ockendon *et.al.* [3].

In Figs 1 and 2 we return to the notation of Section 3. In Fig. 1, $\tilde{p}$ is plotted against $\bar{x}_2$ for a range of values of τ_2. As τ_2 increases, the increase in pressure extends further into the tunnel. In Fig. 2, $\tilde{p}$ is plotted against τ_2 at fixed positions in the tunnel. The pressure $\tilde{p}$ increases steadily form $\tilde{p} = 0$ at $\tau_2 = 0$ to $\tilde{p} = 1$ at $\tau_2 = \infty$.

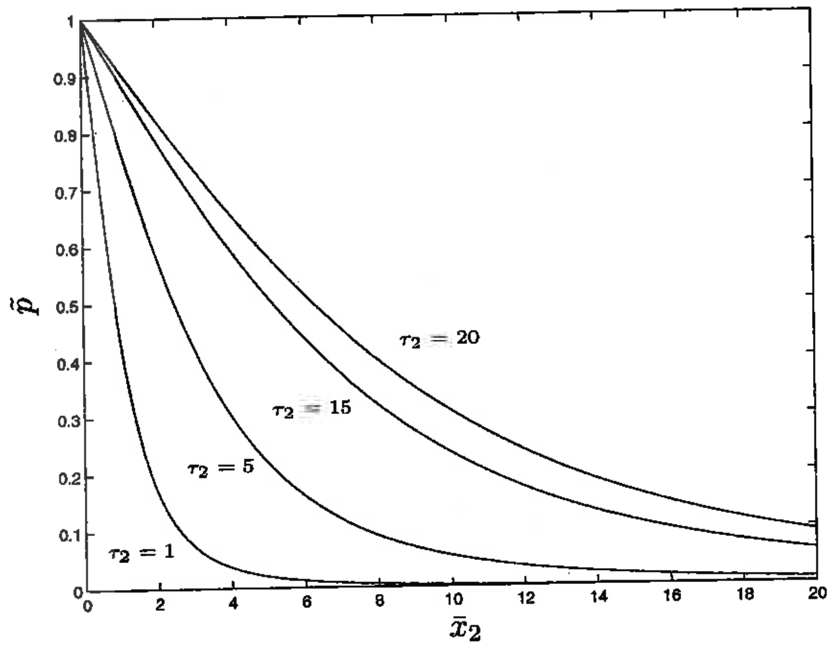

Figure 1: Pressure $\tilde{p}$ given by (4.27) plotted against $\bar{x}_2$ for a range of values of τ_2.

4.2 Diffusion equation for fluid velocity

The fluid velocity u may be calculated directly from the solution (4.27) for $p(t, x)$ using (3.13):

$$\frac{\partial p}{\partial x} = -u^2 . \tag{4.29}$$

Alternatively, u may be derived without prior knowledge of p by solving the nonlinear diffusion equation (4.4), subject to the boundary conditions (4.5) and (4.6). We now consider this second approach in order to illustrate the

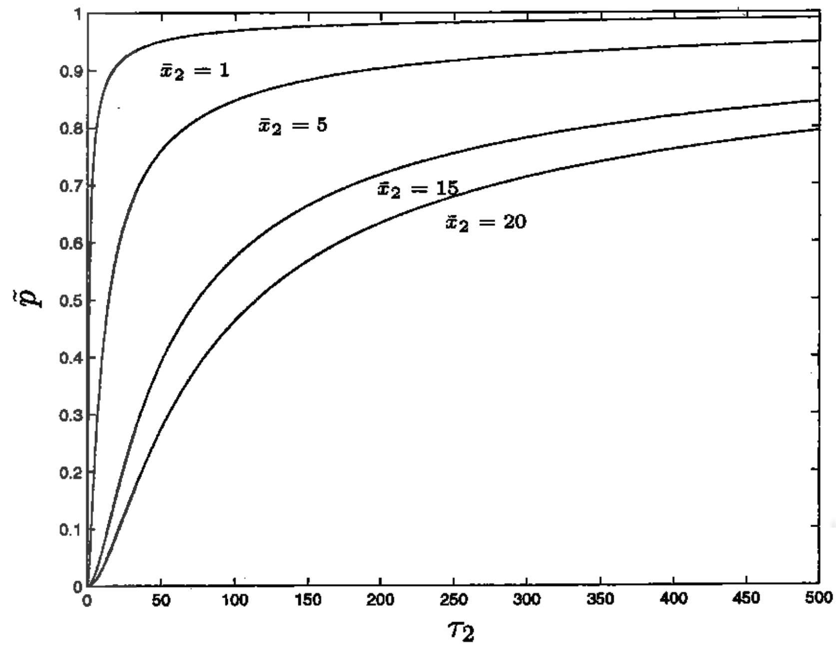

Figure 2: Pressure $\tilde{p}$ given by (4.27) plotted against τ_2 at fixed positions $\bar{x}_2$ in the tunnel.

part played in the solution by a conservation law for the partial differential equation.

By considering the scaling transformation (4.7), with p replaced by u, it can be shown that the similarity solution of (4.4) is of the form

$$u(t,x) = t^{1-2\alpha}F(\xi)\,, \qquad \xi = \frac{x}{t^\alpha} \tag{4.30}$$

where α is a constant. Substituting (4.30) into (4.4) leads to the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + 2\alpha\xi F\,\frac{dF}{d\xi} - 2(1-2\alpha)F^2 = 0\,. \tag{4.31}$$

The boundary conditions (4.5) and (4.6) become

$$t^{1-3\alpha}\frac{dF(0)}{d\xi} = 0\,, \qquad t^{1-2\alpha}\left(F(\xi) - \frac{3}{\xi^2}\right) \sim 0 \quad \text{as} \quad \xi \to \infty\,. \tag{4.32}$$

Clearly, the constant α cannot be determined from the boundary conditions.

To obtain α we first rewrite (4.4) in the form of a conservation law

$$\frac{\partial}{\partial t}(u^2) + \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}\right) = 0\,. \tag{4.33}$$

Integrating (4.33) along the tunnel with respect to x from the entrance $x = 0$ to $x = \infty$ gives

$$\frac{d}{dt}\left(\int_0^\infty u^2(t,x)dx\right) + \left[-\frac{\partial u}{\partial x}(t,x)\right]_{x=0}^{x=\infty} = 0 . \qquad (4.34)$$

But from the boundary conditions (4.5) and (4.6),

$$\frac{\partial u}{\partial x}(t,0) = 0 , \qquad \frac{\partial u}{\partial x}(t,\infty) = 0 \qquad (4.35)$$

and therefore

$$\int_0^\infty u^2(t,x)\, dx = k , \qquad (4.36)$$

where k is a constant independent of t. Substituting (4.30) into (4.36) and changing the variable of integration from x to ξ at fixed time t gives

$$t^{2-3\alpha}\int_0^\infty F^2(\xi)d\xi = k . \qquad (4.37)$$

Hence $\alpha = \frac{2}{3}$. Thus

$$u(t,x) = t^{-1/3}F(\xi) , \qquad \xi = \frac{x}{t^{2/3}} \qquad (4.38)$$

where $F(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + \frac{2}{3}\frac{d}{d\xi}(\xi F^2) = 0 , \qquad (4.39)$$

subject to the boundary conditions

$$\frac{dF}{d\xi}(0) = 0 , \qquad (4.40)$$

$$F(\xi) \sim \frac{3}{\xi^2} \quad \text{as} \quad \xi \to \infty . \qquad (4.41)$$

It remains to obtain the constant k. Integrating (4.29) with respect to x from $x = 0$ to $x = \infty$ and using the boundary conditions (4.2) and (4.3) gives $k = 1$. The conserved quantity therefore is, from (4.37),

$$\int_0^\infty F^2(\xi)d\xi = 1 . \qquad (4.42)$$

Equation (4.39) is readily integrated and imposing the boundary condition (4.40) we find that

$$F(\xi) = \frac{3}{(\xi^2 + a^2)}\,, \tag{4.43}$$

where a is a constant. The boundary condition (4.41) is automatically satisfied. The constant a is obtained from the conserved quantity (4.42). Substituting (4.43) into (4.42) yields

$$a^3 = 9\int_0^\infty \frac{du}{(1+u^2)^2} = \frac{9\pi}{4}\,, \tag{4.44}$$

where the integral was evaluated by letting $u = \tan\theta$. Hence

$$F(\xi) = \frac{3}{\xi^2 + \left(\frac{9\pi}{4}\right)^{2/3}} \tag{4.45}$$

and therefore, from (4.38),

$$u(t,x) = \frac{3t}{x^2 + \left(\frac{9\pi}{4}\,t^2\right)^{2/3}}\,. \tag{4.46}$$

Expressed in the notation of Section 3, (4.46) becomes

$$u = \frac{3\varepsilon^{1/3}\tau_2}{\bar{x}_2^2 + \left(\frac{9\pi}{4}\,\tau_2^2\right)^{2/3}}\,, \tag{4.47}$$

which agrees with the solution of Ockendon *et.al.* [3].

In Fig 3, u is plotted against $\bar{x}_2$ for a range of values of τ_2. At a fixed time τ_2, the velocity u decreases steadily as $\bar{x}_2$ increases.

In Fig. 4, u is plotted against τ_2 for several positions $\bar{x}_2$ in the tunnel. At a given position $\bar{x}_2$, u increases steadily from zero to a maximum value of

$$u_{\max} = 0.643\,\frac{\varepsilon^{1/3}}{\bar{x}_2^{1/2}} \qquad \text{at} \qquad \tau_{2\,\max} = 0857\,\bar{x}_2^{3/2} \tag{4.48}$$

and then decreases steadily to zero as $\tau_2 \to \infty$.

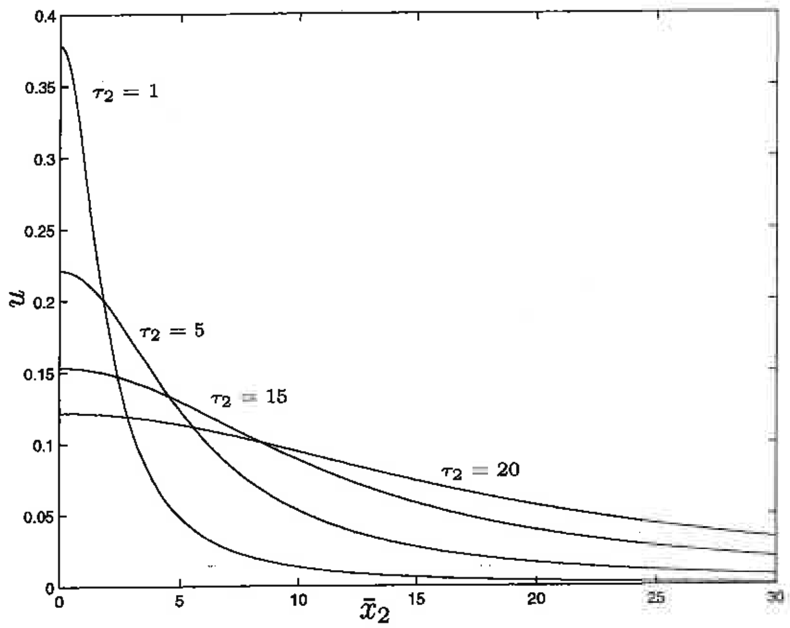

Figure 3: Velocity u given by (4.47) plotted against $\bar{x}_2$ for a range of values of τ_2 with $\epsilon = 10^{-1}$.

4.3 Elementary conservation law for pressure

In Section 4.1 we derived the constant α in the similarity solution (4.13) from the boundary condition $p(t, 0) = 1$. We saw in Section 4.2 that a conserved quantity can also be used to derive the constant α in a similarity solution. In this section we will investigate the application of a conserved quantity to obtain α in (4.13). We will see that one of the boundary conditions (4.2) and (4.3) has to be dropped.

Equation (4.1) can be written in the form of a conservation law as

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left[\left(-\frac{\partial p}{\partial x}\right)^{1/2}\right] = 0 . \qquad (4.49)$$

Integrate (4.49) with respect to x from the tunnel entrance, $x = 0$, to $x = \infty$:

$$\frac{d}{dt}\left(\int_0^\infty p(t, x)dx\right) + \left[\left(-\frac{\partial p}{\partial x}\right)^{1/2}\right]_0^\infty = 0 . \qquad (4.50)$$

From the boundary condition (4.3),

$$\frac{\partial p}{\partial x}(t, \infty) = 0 . \qquad (4.51)$$

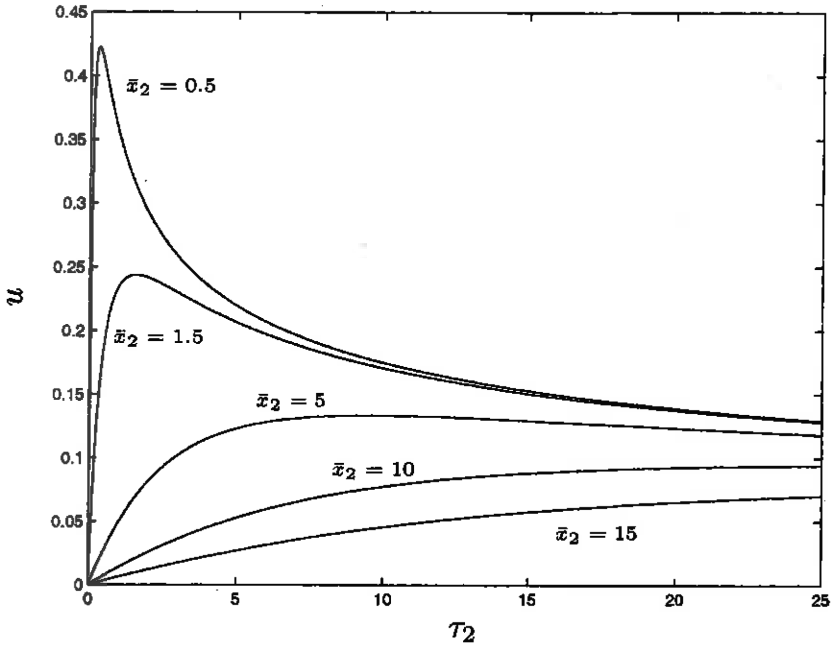

Figure 4: Velocity u given by (4.47) plotted against τ_2 at fixed positions $\bar{x}_2$ in the tunnel with $\varepsilon = 10^{-1}$.

However, the boundary condition

$$\frac{\partial p}{\partial x}(t, 0) = 0 \tag{4.52}$$

is not in general satisfied. We will look for a solution for which (4.52) holds and compare the results with the solution derived in Section 4.1. If (4.52) is satisfied then

$$\frac{d}{dt}\left(\int_0^\infty p(t, x)dx\right) = 0 \tag{4.53}$$

and therefore

$$\int_0^\infty p(t, x)dx = P\ , \tag{4.54}$$

where P is independent of t and therefore a constant, which represents the strength of the pressure pulse.

Substitute (4.13) into (4.54) and make the change of variable from x to ξ at fixed time t. Equation (4.54) becomes

$$t^{2-2\alpha}\int_0^\infty F(\xi)d\xi = P \tag{4.55}$$

and hence $\alpha = 1$. Thus from (4.13),

$$p(t,x) = \frac{1}{t} F(\xi) , \qquad \xi = \frac{x}{t} , \tag{4.56}$$

where from (4.14), $F(\xi)$ satisfies the differential equation

$$\frac{d^2F}{d\xi^2} + 2\left(-\frac{dF}{d\xi}\right)^{1/2} \frac{d}{d\xi}(\xi F) = 0 , \tag{4.57}$$

subject to the boundary conditions (4.52) and (4.3),

$$\frac{dF}{d\xi}(0) = 0 , \tag{4.58}$$

$$F(\xi) \sim \frac{3}{\xi^3} \qquad \text{as} \qquad \xi \to \infty . \tag{4.59}$$

Clearly, the boundary condition (4.2), $p(t,0) = 1$, is not satisfied by (4.56). The conserved quantity (4.55) becomes

$$\int_0^\infty F(\xi) d\xi = P . \tag{4.60}$$

The differential equation (4.57) can be written as

$$\frac{d}{d\xi}\left[\left(-\frac{dF}{d\xi}\right)^{1/2}\right] - \frac{d}{d\xi}(\xi F) = 0 . \tag{4.61}$$

Unlike (4.17), (4.57) is not a first order differential equation in $\frac{dF}{d\xi}$. The solution of (4.61) subject to the boundary condition (4.58) is

$$F(\xi) = \frac{3}{c^3 + \xi^3} , \tag{4.62}$$

where $c > 0$ is a constant. The boundary condition (4.59) is automatically satisfied by (4.62). The constant c is obtained by substituting (4.62) into the conserved quantity (4.60). This gives

$$c = \left[\frac{3}{P}\int_0^\infty \frac{d\eta}{1+\eta^3}\right]^{1/2} = \frac{1.9046}{\sqrt{P}} , \tag{4.63}$$

since [7]

$$\int_0^\infty \frac{d\eta}{1+\eta^3} = \frac{2\pi}{3\sqrt{3}} . \tag{4.64}$$

From (4.56),

$$p(t,x) = \frac{3t^2}{x^3 + (ct)^3} \,. \tag{4.65}$$

Hence, at $x = 0$,

$$p(t,0) = \frac{3}{c^3 t} = 0.434 \, \frac{P^{3/2}}{t} \,. \tag{4.66}$$

The boundary condition $p(t,0) = 1$ is not satisfied. The initial pressure at the tunnel entrance is infinite but the strength of the air blast defined by P is finite. The perturbation theory developed in Section 3 would break down for small values of time near the tunnel entrance as well as for large values of P.

In Figs 5, 6 and 7 we use the notation of Section 3. In Fig. 5, $\tilde{p}$ is plotted against $\bar{x}_2$ for a range of values of τ_2. The pressure decreases steadily as τ_2 increases. It extends an increasing distance into the tunnel as τ_2 increases. In Fig. 6, $\tilde{p}$ is plotted against τ_2 at several fixed points $\bar{x}_2$ in the tunnel. At a given point $\bar{x}_2$, $\tilde{p}$ grows steadily from zero to the maximum value

$$\tilde{p}_{\max} = 0.437 \, \frac{P}{\bar{x}_2} \quad \text{at} \quad \tau_{2\,\max} = 0.662 \, \bar{x}_2 \sqrt{P} \tag{4.67}$$

and then decreases steadily to zero as $\tau_2 \to \infty$. As we move further into the tunnel, corresponding to larger values of x_2, the maximum pressure decreases and occurs at a later time. In Fig. 7, $\tilde{p}$ is plotted against τ_2 at a fixed position $\bar{x}_2$ for a range of values of P. As P increases the maximum pressure increases and occurs at a later time. We see clearly that P is a measure of the strength of the pressure pulse.

5 Conservation laws

Equations (4.33) and (4.49) are examples of conservation laws. We saw in Section 4 the important part conservation laws could play in obtaining similarity solutions of partial differential equations. In this section we investigate if other conservation laws exist for the partial differential equations obtained in Section 3.

5.1 Nonlinear diffusion equation for pressure

Consider the nonlinear diffusion equation (4.1) for the pressure. A conservation law for equation (4.1) is of the form [8]

$$D_1 T^1 + D_2 T^2 \Big|_{(4.1)} = 0 \,, \tag{5.1}$$

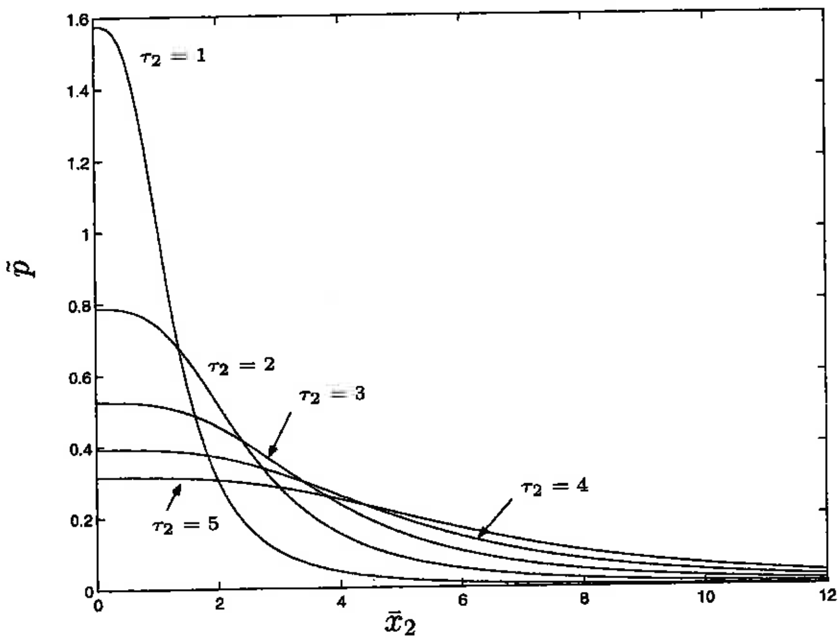

Figure 5: Pressure $\tilde{p}$ given by (4.65) plotted against $\bar{x}_2$ for a range of values of τ_2 with $P = 1$.

where the total derivatives, D_1 and D_2, are defined by

$$D_1 = D_t = \frac{\partial}{\partial t} + p_t \frac{\partial}{\partial p} + p_{tt} \frac{\partial}{\partial p_t} + p_{xt} \frac{\partial}{\partial p_x} + \cdots , \tag{5.2}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + p_x \frac{\partial}{\partial p} + p_{tx} \frac{\partial}{\partial p_t} + p_{xx} \frac{\partial}{\partial p_x} + \cdots \tag{5.3}$$

and subscripts denote partial derivatives. The vector $T = (T^1, T^2)$ is a conserved vector for the partial differential equation (4.1). Equation (4.49) can be written in the form

$$D_1(p) + D_2\left((-p_x)^{1/2}\right)\Big|_{4.1} = 0 . \tag{5.4}$$

The conserved vector

$$T = \left(p, (-p_x)^{1/2}\right) \tag{5.5}$$

is referred to as the elementary conserved vector for (4.1).

We will look for conserved vectors of the form

$$T^1 = T^1(t, x, p, p_t) , \qquad T^2 = T^2(t, x, p, p_x) . \tag{5.6}$$

More general forms for T^1 and T^2 could be considered but the calculations become progressively more difficult as additional variables are added to T^1

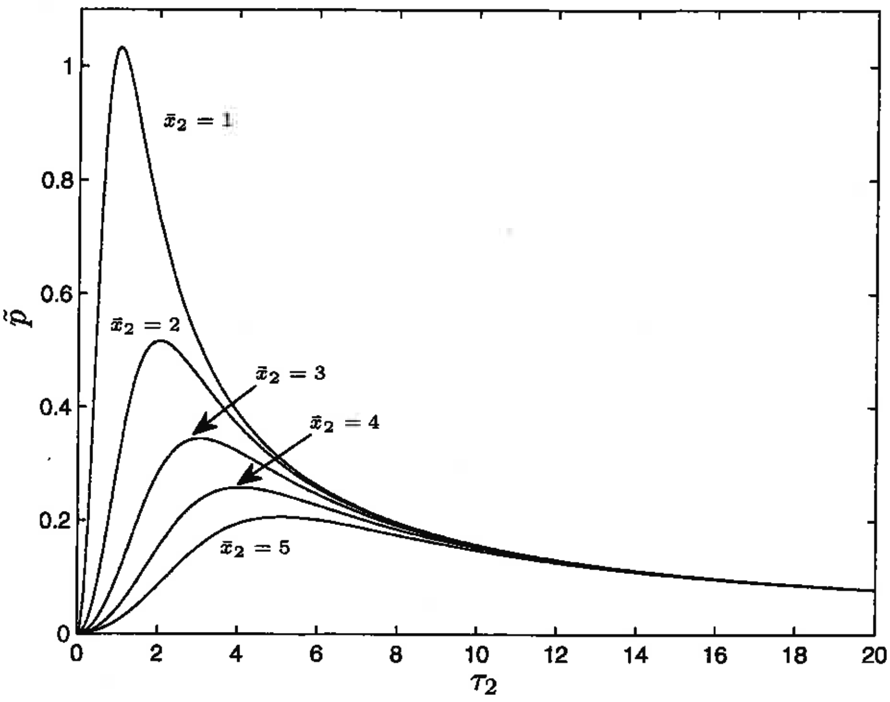

Figure 6: Pressure $\tilde{p}$ given by (4.65) plotted against τ_2 at fixed position $\bar{x}_2$ in the tunnel when P=1.

and T^2. We will use the direct method for deriving conservation laws which consists in substituting (5.6) into (5.4) and separating by powers of the variables on which T^1 and T^2 do not depend. Substitute (5.6) into (5.1) and replace p_{xx} using (4.1):

$$p_{xx} = 2(-p_x)^{1/2} p_{xx} \, . \tag{5.7}$$

This gives

$$\frac{\partial T^1}{\partial t} + p_t \frac{\partial T^1}{\partial p} + p_{tt} \frac{\partial T^1}{\partial p_t} + \frac{\partial T^2}{\partial x} + p_x \frac{\partial T^2}{\partial p} + 2(-p_x)^{1/2} p_t \frac{\partial T^2}{\partial p_x} = 0 \, . \tag{5.8}$$

Since T^1 and T^2 do not depend on p_{tt} we can separate (5.8) by powers of p_{tt}:

$$p_{tt} : \qquad \frac{\partial T^1}{\partial p_t} = 0, \tag{5.9}$$

$$\text{remainder:} \; \frac{\partial T^1}{\partial t} + p_t \frac{\partial T^1}{\partial p} + \frac{\partial T^2}{\partial x} + p_x \frac{\partial T^2}{\partial p} + 2(-p_x)^{1/2} p_t \frac{\partial T^2}{\partial p_x} = 0. \tag{5.10}$$

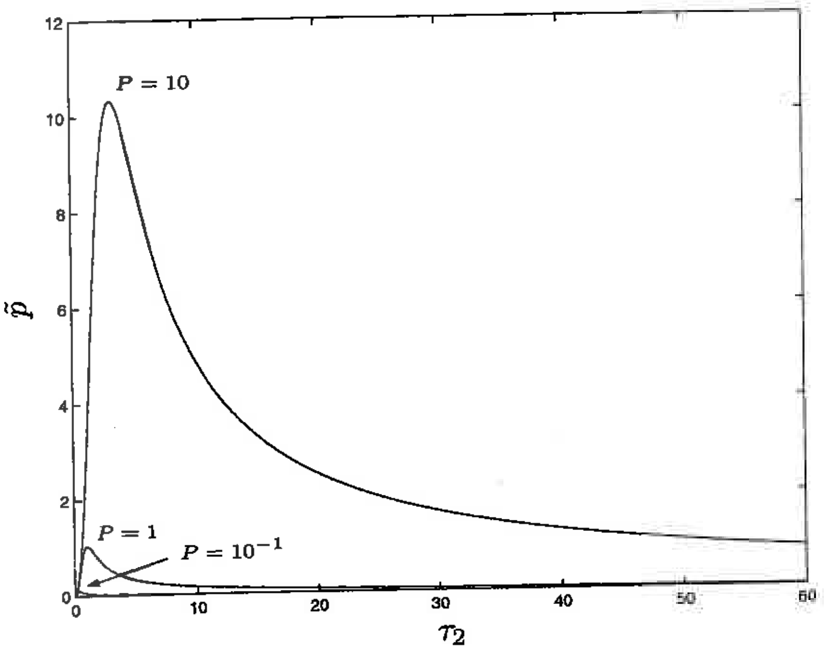

Figure 7: Pressure $\tilde{p}$ given by (4.65) plotted against τ_2 at $\bar{x}_2 = 1$ for a range of values of P.

From (5.9), $T^1 = T^1(t,x,p)$ and we can therefore separate (5.10) according to powers of p_t:

$$p_t : \qquad \frac{\partial T^1}{\partial t} + 2(-p_x)^{1/2}\frac{\partial T^2}{\partial p_x} = 0, \tag{5.11}$$

$$\text{remainder:} \quad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + p_x\frac{\partial T^2}{\partial p} = 0. \tag{5.12}$$

From (5.11),

$$T^2(t,x,p,p_x) = (-p_x)^{1/2}\frac{\partial T^1}{\partial p}(t,x,p) + A(t,x,p)\,. \tag{5.13}$$

Substitute (5.13) into (5.12) and separate according to powers of p_x:

$$(-p_x)^{3/2} : \quad \frac{\partial^2 T^1}{\partial p^2}(t,x,p) = 0, \tag{5.14}$$

$$p_x : \quad \frac{\partial A}{\partial p}(t,x,p) = 0, \tag{5.15}$$

$$(-p_x)^{1/2}: \quad \frac{\partial^2 T^1}{\partial x \partial p}(t,x,p) = 0, \tag{5.16}$$

$$\text{remainder:} \quad \frac{\partial T^1}{\partial t}(t,x,p) + \frac{\partial A}{\partial x}(t,x,p) = 0. \tag{5.17}$$

Thus from (5.15), $A = A(t,x)$ and from (5.14)and (5.16)

$$T^1(t,x,p) = p\,B(t) + C(t,x)\,. \tag{5.18}$$

Substituting (5.18) into (5.17) gives

$$p\frac{dB(t)}{dt} + \frac{\partial C(t,x)}{\partial t} + \frac{\partial A(t,x)}{\partial x} = 0\,. \tag{5.19}$$

Separating (5.19) by powers of p gives $B(t) = B_0$ and

$$\frac{\partial C(t,x)}{\partial t} + \frac{\partial A(t,x)}{\partial x} = 0\,. \tag{5.20}$$

Hence, from (5.18 and (5.13),

$$T^1(t,x,p) = B_0 p + C(t,x)\,, \tag{5.21}$$

$$T^2(t,x,p_x) = B_0(-p_x)^{1/2} + A(t,x)\,, \tag{5.22}$$

where $C(t,x)$ and $A(t,x)$ satisfy (5.20). Now the conserved vector

$$T^1 = C(t,x)\,, \qquad T^2 = A(t,x)\,, \tag{5.23}$$

where $C(t,x)$ and $A(t,x)$ satisfy (5.20) is a trivial conserved vector because

$$D_1 T^1 + D_2 T^2 = \frac{\partial C}{\partial t} + \frac{\partial A}{\partial x} \equiv 0 \tag{5.24}$$

without imposing the condition that (4.1) is satisfied. Thus the only conserved vector of the form (5.6) for the partial differential equation (4.1) is the elementary conserved vector (5.5). There may of course be conserved vectors for (4.1) more general than (5.5).

5.2 Nonlinear diffusion equation for velocity

We saw in Section 4.2 the important part played by the conservation law (4.43) in the similarity solution of the nonlinear diffusion equation (4.4). We now investigate if there are other conservation laws for equation (4.4).

A conservation law for (4.4) satisfies

$$D_1T^1 + D_2T^2\Big|_{(4.4)} = 0\,, \tag{5.25}$$

where the total derivatives, D_1 and D_2, are now defined by

$$D_1 = D_t = \frac{\partial}{\partial t} + u_t\,\frac{\partial}{\partial u} + u_{tt}\,\frac{\partial}{\partial u_t} + u_{xt}\,\frac{\partial}{\partial u_x} + \cdots\,, \tag{5.26}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + u_x\,\frac{\partial}{\partial u} + u_{tx}\,\frac{\partial}{\partial u_t} + u_{xx}\,\frac{\partial}{\partial u_x} + \cdots\,. \tag{5.27}$$

Equation (4.33) can be written as

$$D_1(u^2) + D_2(-u_x)\Big|_{(4.4)} = 0\,. \tag{5.28}$$

The conserved vector

$$T = (u^2, -u_x) \tag{5.29}$$

is the elementary conserved vector for (4.4).

Consider conserved vectors for (4.4) of the form

$$T^1 = T^1(t,x,u,u_t)\,, \qquad T^2 = T^2(t,x,u,u_x)\,. \tag{5.30}$$

Substitute (5.30) into (5.25) and replace u_{xx} using (4.4),

$$u_{xx} = 2uu_t\,. \tag{5.31}$$

We obtain

$$\frac{\partial T^1}{\partial t} + u_t\,\frac{\partial T^1}{\partial u} + u_{tt}\,\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} + 2uu_t\,\frac{\partial T^2}{\partial u_x} = 0\,. \tag{5.32}$$

Now T^1 and T^2 are independent of u_{tt} and we can therefore separate (5.32) according to powers of u_{tt}:

$$u_{tt}: \qquad \frac{\partial T^1}{\partial u_t} = 0, \tag{5.33}$$

$$\text{remainder:} \qquad \frac{\partial T^1}{\partial t} + u_t\,\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} + 2uu_t\,\frac{\partial T^2}{\partial u_x} = 0. \tag{5.34}$$

Thus from (5.33) $T^1 = T^1(t,x,u)$ and since T^1 and T^2 do not depend on u_t, (5.34) can be separated by u_t to give

$$u_t: \qquad \frac{\partial T^1}{\partial u} + 2u\,\frac{\partial T^2}{\partial u_x} = 0, \tag{5.35}$$

$$\text{remainder:} \qquad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} = 0. \tag{5.36}$$

Differentiating (5.35) with respect to u_x, we obtain

$$\frac{\partial^2 T^2}{\partial u_x^2} = 0 \tag{5.37}$$

and therefore

$$T^2(t,x,u,u_x) = u_x\,A(t,x,u) + B(t,x,u)\,. \tag{5.38}$$

Substituting (5.38) back into (5.35) we obtain

$$\frac{\partial T^1}{\partial u} + 2u\,A(t,x,u) = 0\,. \tag{5.39}$$

It remains to find $A(t,x,u)$ and $B(t,x,u)$. Substitute (5.38) into (5.36) and separate the resulting equation by powers of u_x to obtain

$$u_x^2: \qquad \frac{\partial A}{\partial u}\,(t,x,u) = 0, \tag{5.40}$$

$$u_x: \qquad \frac{\partial A}{\partial x}\,(t,x,u) + \frac{\partial B}{\partial u}\,(t,x,u) = 0, \tag{5.41}$$

$$\text{remainder:} \qquad \frac{\partial T^1}{\partial t}\,(t,x,u) + \frac{\partial B}{\partial x}\,(t,x,u) = 0. \tag{5.42}$$

From (5.40), $A = A(t,x)$ and therefore from (5.41)

$$B(t,x,u) = -u\frac{\partial A(t,x)}{\partial x} + C(t,x) \tag{5.43}$$

and from (5.39)

$$T^1(t,x,u) = -u^2 A(t,x) + D(t,x)\,. \tag{5.45}$$

By substituting (5.43) and (5.44) into (5.42) and separating by powers of u we obtain

$$u^2: \qquad \frac{\partial A}{\partial t}\,(t,x) = 0, \tag{5.45}$$

$$u: \qquad \frac{\partial^2 A(t,x)}{\partial x^2} = 0, \tag{5.46}$$

$$\text{remainder:} \qquad \frac{\partial D(t,x)}{\partial t} + \frac{\partial C(t,x)}{\partial x} = 0. \tag{5.47}$$

It follows from (5.45) and (5.46) that

$$A(x) = A_1 x + A_2 \,, \tag{5.48}$$

where A_1 and A_2 are constants and hence from (5.43),

$$B(t,x,u) = -A_1 u + C(t,x) \,. \tag{5.49}$$

Let $A_1 = -c_2$ and $A_2 = -c_1$. Then (5.44) and (5.38) become

$$T^1(t,x,u) = c_1 u^2 + c_2 x u^2 + D(t,x) \,, \tag{5.50}$$

$$T^2(t,x,u,u_x) = c_1(-u_x) + c_2(u - x u_x) + C(t,x) \,, \tag{5.51}$$

where $D(t,x)$ and $C(t,x)$ satisfy (5.47). The conserved vector

$$T^1 = D(t,x) \,, \qquad T^2 = C(t,x) \,, \tag{5.52}$$

is a trivial conserved vector because the conservation law (5.25) is identically satisfied. The conserved vector is therefore a linear combination of two conserved vectors.

$$T^1 = u^2 \,, \qquad T^2 = -u_x \,, \tag{5.53}$$

$$T^1 = x u^2 \,, \qquad T^2 = u - x u_x \,. \tag{5.54}$$

Equation (5.54) is the elementary conserved vector. The conserved vector (5.54) is new.

5.3 Nonlinear wave equation

To simplify the notation we will write (3.8) as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -2u \frac{\partial u}{\partial t} \tag{5,55}$$

and the boundary conditions (3.9) and (3.10) as

$$x = t: \qquad u = \frac{2}{t}, \tag{5.56}$$

$$x \to 0: \qquad u \sim \frac{3t}{x^2}. \tag{5.57}$$

Conserved vectors for (5.55) satisfy

$$D_1 T^1 + D_2 T^2\Big|_{(5.55)} = 0\,, \tag{5.58}$$

where D_1 and D_2 are defined by (5.26) and (5.27). We will look for conserved vectors of the form (5.30). Substitute (5.30) into (5.58) and replace u_{xx} by

$$u_{xx} = u_{tt} + 2u\,u_t\,. \tag{5.59}$$

This yields

$$\frac{\partial T^1}{\partial t} + u_t\,\frac{\partial T^1}{\partial u} + u_{tt}\,\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} + (u_{tt} + 2u\,u_t)\,\frac{\partial T^2}{\partial u_x} = 0\,. \tag{5.60}$$

Since T^1 and T^2 are independent of u_{tt} we separate (5.60) by powers of u_{tt}:

u_{tt} :
$$\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \tag{5.61}$$

remainder:
$$\frac{\partial T^1}{\partial t} + u_t\,\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\,\frac{\partial T^2}{\partial u} + 2u\,u_t\,\frac{\partial T^2}{\partial u_x} = 0. \tag{5.62}$$

Differentiating (5.61 with respect to u_t and then integrating twice with respect to u_t gives

$$T^1(t,x,u,u_t) = u_t\,A(t,x,u) + B(t,x,u)\,. \tag{5.63}$$

Substituting (5.63) back into (5.61) and integrating with respect to u_x gives

$$T^2(t,x,u,u_x) = -u_x\,A(t,x,u) + C(t,x,u)\,. \tag{5.64}$$

It remains to obtain $A(t,x,u)$, $B(t,x,u)$ and $C(t,x,u)$. Substitute (5.63) and (5.64) into (5.62) and separate by powers of u_t and u_x:

u_t^2 :
$$\frac{\partial A}{\partial u} = 0, \tag{5.65}$$

u_x^2 :
$$\frac{\partial A}{\partial u} = 0, \tag{5.66}$$

u_t :
$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial u} - 2u\,A = 0, \tag{5.67}$$

u_x :
$$\frac{\partial A}{\partial x} - \frac{\partial C}{\partial u} = 0, \tag{5.68}$$

$$\text{remainder:} \qquad \frac{\partial B}{\partial t} + \frac{\partial C}{\partial x} = 0. \tag{5.69}$$

From (5.65) and (5.66), $A = A(t,x)$ and solving (5.67) for $B(t,x,u)$ we obtain

$$B(t,x,u) = u^2 A(t,x) - u\,\frac{\partial A(t,x)}{\partial x} + D(t,x)\,. \tag{5.70}$$

To obtain $C(t,x,u)$ we integrate (5.68) with respect to u:

$$C(t,x,u) = u\,\frac{\partial A(t,x)}{\partial x} + E(t,x)\,. \tag{5.71}$$

Finally, we substitute (5.70) and (5.71) into (5.69) and separate by powers of u:

$$u^2: \qquad \frac{\partial A(t,x)}{\partial t} = 0, \tag{5.72}$$

$$u: \qquad \frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = 0, \tag{5.73}$$

$$\text{remainder:} \qquad \frac{\partial D}{\partial t} + \frac{\partial E}{\partial x} = 0. \tag{5.74}$$

From (5.72) and (5.73),

$$A(x) = c_1 + c_2 x\,, \tag{5.75}$$

where c_1 and c_2 are constants and therefore from (5.70) and (5.71),

$$B(t,x,u) = (c_1 + c_2 x)u^2 + D(t,x)\,, \tag{5.76}$$

$$C(t,x,u) = c_2 u + E(t,x)\,. \tag{5.77}$$

Equations (5.63) and (5.64) become

$$T^1 = c_1(u^2 + u_t) + c_2(xu^2 + xu_t) + D(t,x)\,, \tag{5.78}$$

$$T^2 = c_1(-u_x) + c_2(u - xu_x) + E(t,x)\,, \tag{5.79}$$

where $D(t,x)$ and $E(t,x)$ satisfy (5.74). The conservation law

$$T^1 = D(t,x)\,, \qquad T^2 = E(t,x)\,, \tag{5.80}$$

is a trivial conservation law. Non-trivial conserved vectors of the form (5.30) are therefore linear combinations of the two conserved vectors

$$T^1 = u^2 + u_t\,, \qquad T^2 = -u_x\,, \tag{5.81}$$

Partial differential equation	Conserved vectors
$p_t = \frac{1}{2(-p_x)^{1/2}} p_{xx}$	$T^1 = p\,, \qquad T^2 = (-p_x)^{1/2}$
$u_{xx} = 2uu_t$	$T^1 = u^2\,, \quad T^2 = -u_x$ $T^1 = xu^2\,, \quad T^2 = u - xu_x$
$u_{tt} - u_{xx} = -2uu_t$	$T^1 = u^2 + u_t\,, \qquad T^2 = -u_x$ $T^1 = x(u^2 + u_t)\,, \quad T^2 = u - xu_x$

Table 1: Conservation laws

$$T^1 = x(u^2 + u_t)\,, \qquad T^2 = u - xu_x\,. \tag{5.82}$$

The conserved vector (5.81) is readily derived by writing (5.55) as

$$\frac{\partial}{\partial t}(u^2 + u_t) + \frac{\partial}{\partial x}(-u_x) = 0\,. \tag{5.83}$$

It is the elementary conserved vector.

The derivation of the conserved vectors for a partial differential equation does not depend on the boundary conditions. The derivation of a conserved quantity from the conservation law does depend on the boundary conditions through the component T^2.

The conservation laws derived for the two nonlinear diffusion equations, (4.1) and (4.4), and for the nonlinear wave equation, (5.55), are summarised in Table 1

6 Compressive wave due to motion of plug into tunnel

We now consider the turbulent compressible flow due to the plug moving into the tunnel as a result of the air blast. This is modelled by Fanno flow driven by a piston which was also considered by Ockendon *et.al.* [3]

Consider a piston moved impulsively with constant velocity u_0 such that $u_0 << a_0$ where a_0 is the speed of sound in the gas at rest. The small parameter in the flow is $\varepsilon = u_0/a_0$. The same scalings as used for the pressure wave are employed and τ is defined by $t = \varepsilon\tau$. The equation of the piston in the (x, τ) plane is $x = \varepsilon\tau$ and the equation of the shock is again given by (3.2). The boundary condition at the piston is

$$x = \varepsilon\tau : \qquad u = 1 . \tag{6.1}$$

Consider time $\tau = \mathrm{O}(\varepsilon^{-2})$. For the region near the shock, $x = \mathrm{O}(\varepsilon^{-2})$ and equations (3.5) to (3.7) again apply. But $u_2 \to \infty$ as $x_2 \to \infty$ and an inner region closer to the piston has to be introduced to satisfy the boundary condition (6.1) at the piston. By considering how u_2 grows as $x_2 \to 0$, Ockendon *et.al.* [3] defined the inner variable $\hat{x}_2$ by

$$x_2 = \varepsilon^{1/2}\hat{x}_2 . \tag{6.2}$$

In this region $x = \mathrm{O}(\varepsilon^{-3/2})$ which is closer to the piston than $x = \mathrm{O}(\varepsilon^{-2})$. In equations (3.5) to (3.7) we make the transformation of independent variables from (x_2, τ_2) to $(\hat{x}_2, \tau_2)$ and dependent variables from (u_2, p_2, ρ_2) to $(u, \tilde{p}, \tilde{\rho})$ where for this problem,

$$\tilde{p} = \mathrm{O}(\varepsilon^{-1/2}) , \quad \tilde{\rho} = \mathrm{O}(\varepsilon^{-1/2}) , \quad u = 0(1) . \tag{6.3}$$

Equations (3.5) to (3.7) become

$$\varepsilon^{1/2}\frac{\partial\tilde{\rho}}{\partial\tau_2} + \frac{\partial u}{\partial\hat{x}_2} = 0 , \tag{6.4}$$

$$\varepsilon^{1/2}\frac{\partial\tilde{p}}{\partial\hat{x}_2} = -u^2 , \tag{6.5}$$

$$\varepsilon^{1/2}\frac{\partial\tilde{p}}{\partial\tau_2} + \frac{\partial u}{\partial\hat{x}_2} = 0 . \tag{6.6}$$

Since the boundary conditions on u are known we eliminate $\tilde{p}$ from (6.5) and (6.6). This gives the nonlinear diffusion equation for u,

$$\frac{\partial^2 u}{\partial\hat{x}_2^2} = 2u\,\frac{\partial u}{\partial\tau_2} . \tag{6.7}$$

The boundary conditions are

$$\hat{x}_2 = 0 : \qquad u = 1 , \tag{6.8}$$

$$\hat{x}_2 \to \infty : \qquad u \sim 3\,\frac{\tau_2}{\hat{x}_2^2} . \tag{6.9}$$

Using a similarity transformation Ockendon *et.al.* [3] reduced (6.7) to an ordinary differential equation and performed a Lie plane analysis [5]. We will investigate the conservation laws for (6.7).

7 Conservation laws for compressive wave

In order to simplify notation we write (6.7) as

$$\frac{\partial^2 u}{\partial x^2} = 2u\,\frac{\partial u}{\partial t} \tag{7.1}$$

and the boundary conditions (6.8) and (6.9) as

$$x = 0: \qquad u = 1\,, \tag{7.2}$$

$$x \to \infty: \qquad u \sim \frac{3t}{x^2}\,. \tag{7.3}$$

The compressive piston problem, (7.1) to (7.3), differs form the problem of small amplitude waves, (4.4) to (4.6), by the boundary conditions, (7.2) and (4.5).

Conservation laws do not depend on boundary conditions. Two conserved vectors for (7.1), given by (5.53) and (5.54), were derived in Section 5.2. The elementary conserved vector (5.54) lead to a conserved quantity for small amplitude waves because of the boundary condition $u_x(t, 0) = 0$. Consider the second conserved vector (5.54). The corresponding conservation law can be written as

$$\frac{\partial}{\partial t}\,(xu^2) + \frac{\partial}{\partial x}\,(u - xu_x) = 0 \tag{7.4}$$

and integrating with respect to x from $x = 0$ to $x = \infty$ we obtain

$$\frac{d}{dt}\left(\int_0^\infty xu^2(t,x)dx\right) + \Big[u - xu_x\Big]_0^\infty = 0\,. \tag{7.5}$$

Using the boundary conditions (7.2) and (7.3) and assuming that $u_x(t, 0)$ is finite, we obtain

$$\frac{d}{dt}\left(\int_0^\infty xu^2(t,x)dx\right) = 1\,. \tag{7.6}$$

The similarity solution for equation (7.1), given by (4.30), is independent of the boundary conditions and contains a parameter α. For small amplitude waves we found using the conserved vector (5.53) that $\alpha = \frac{2}{3}$. For compressive waves, α can be determined most easily from boundary condition (7.2). Substituting (4.30) into (7.2) gives

$$t^{1-2\alpha}F(0) = 1 \tag{7.7}$$

and therefore $\alpha = \frac{1}{2}$. Thus from (4.30),

$$u(t,x) = F(\xi) \, , \qquad \xi = \frac{x}{t^{1/2}} \tag{7.8}$$

and substituting (7.8) into (7.6) we obtain conserved quantity

$$\int_0^\infty \xi F^2(\xi) d\xi = 1 \, . \tag{7.9}$$

Results of the form (7.9) may be useful when checking the accuracy of numerical solutions.

8 Conclusions

The Fanno model is applicable if the flow is turbulent and the tunnel is long enough for wall drag to be important. In a mining environment the diameter of a tunnel may be 4 m and for an aspect ratio of 10^{-3} the tunnel network would have to be about 4 km in length. This may be possible in mining situations where there are interconnections by tunnels and shafts between underground excavations.

We found conservation laws for the asymptotic reductions of the Fanno model. We derived two linearly independent conservation laws for the nonlinear diffusion equation for the velocity and for the nonlinear wave equation. In both cases one conservation law is the elementary conservation law which is readily derived from the differential equation while the second conservation law is new and is not immediately obvious. For the nonlinear diffusion equation for the pressure we found only one conservation law, the elementary conservation law, even although the form of the conserved vector at the start of the analysis was the same as for the other two partial differential equations. The conservation laws do not depend on the boundary conditions. However, the conserved quantities derived from the conservation laws depend critically on the boundary conditions.

The conservation laws and the conserved quantities derived from them were useful when investigating the asymptotic reductions of the Fanno model. A conserved quantity can be used to determine the unknown parameter in a similarity solution. It may also be useful when checking the accuracy of a numerical solution.

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