# SUPPORT TO ROCK EXCAVATIONS PROVIDED BY SPRAYED LINERS

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**Abstract**

The support provided to mining excavations by sprayed liners is investigated. A cylindrical excavation of circular cross-section is considered. Three models are analysed. In the first model the liner forms a thin cylindrical layer firmly bonded to the inner surface of the excavation. In the second and third models the liner material penetrates into open joints and fractures in the rock. In the second model the liner material penetrates into the fractures to their full extent. In the third model the liner material penetrates some distance into the fractures but the fractures extend well beyond this depth and are taken to extend to infinity. The excavation is subjected to a perturbation in the form of a uniform shear at infinity. Two cylindrical regions are considered, an inner region and an outer region which extends to infinity. The stress and displacement in each region are calculated using plane strain theory of elasticity. In the first model the transfer of stress from the rock to the liner is investigated and the decrease in the stress concentration factor in the rock due to the liner is calculated. In the second and third models the change in the displacement, rotation and stress due to the penetration of the liner material into joints and fractures is determined.

## 1 Introduction

Shotcrete, or pneumatically sprayed concrete, has been used for support of mining excavations in rock for about fifty years. The thickness of a typical shotcrete layer is about 50mm. The design of this support is usually based

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on the assumption that the liner acts as an arch. It is known that thin layers of shotcrete, 10 to 20mm thick, also provide significant support. Recently the use of thin spray-on liners, typically 4mm thick or less, for rock support has increased in the mining industry. Unlike shotcrete, they are usually very flexible and hence their structural capacity is negligible. Although no arch action can be claimed the support given to rock masses by thin spray-on liners is often better than expected.

Liners are considered to be thin if their thickness is less than 20mm. There is no suitable design method for thin liners or analysis of their behaviour that can take into account the numerous possible support mechanisms and their relative contributions. One factor that is seen to be very important is the contribution of the liner to the prevention of the loosening of the rock mass.

A review of support mechanisms provided by liners has been given by Stacey [7]. Stacey and Yu [8] have investigated mechanisms of rock support provided by sprayed liners using numerical modelling. Wang and Tannant [10] used discrete element models to simulate numerically a liner round a tunnel. Recent progress in research on surface support in mining with a particular emphasis on thin spray-on liners is contained in the book edited by Potvin, Stacey and Hadjigeorgiou [4]. We will consider here three simple analytical models to investigate mechanisms of support provided by thin liners.

We will consider a cylindrical excavation of circular cross-section. The excavation will be subjected to a perturbation in the form of a shear stress. This could model the change in stress acting on the excavation as a result of the mining or due to a geological disturbance. The effect of the shear perturbation on an excavation without the application of the liner material will be compared with its effect on an excavation to which the liner material has been applied. Three models will be considered. The three models use the same mathematical results but interpreted in a different way.

In the first model the liner forms a thin cylindrical layer firmly bonded to the inner surface of the excavation. The transfer of stress from the rock to the liner will be investigated and the stress concentration factor [2] at the surface of the excavation will be calculated with and without the liner. The effect of the liner on the displacement and rotation in the rock will be investigated. This could model a thin layer of shotcrete that does not penetrate to a significant depth into fractures at the surface of the excavation. Support mechanisms that do not depend on penetration of liner material into fractures such as the thickness, stiffness and tensile strength of the liner can be investigated. This model would be particularly applicable to a mechanically bored excavation in which the excavation process has not disturbed the rock or caused fractures to develop in the rock. The only planes along which the liner material could penetrate are the natural planes of weakness.

In the second and third models the liner material penetrates into open joints and fractures in the rock. The penetration of the liner material inhibits the movement of the blocks of rock. It is relevant in all jointed rock situations but especially in regions of very high stress such as at the surface of an excavation in which loosening and stress fracturing have taken place. The thin layer of liner at the surface of the excavation will be neglected. The effect of the liner is modelled by the layer of fractured rock penetrated by the liner material. Before the application of the liner material this layer of fractured rock will have Young's modulus and Poisson ratio less than that of the unfractured rock [3]. After the application of the liner material it will be assumed that the Young's modulus and Poisson ratio of the fractured rock penetrated by the liner material are the same as for the unfractured rock.

In the second model it is assumed that the liner material penetrates into the fractures to their full extent. The rock outside of the liner penetration contains no fractures, only planes of weakness. This model would be representative of excavation by careful blasting with any loose material having been scaled or barred down. The depth of penetration would be typically 100 to 200mm. In the third model the liner material penetrates some distance into the fractures but the fractures extend well beyond this depth. This model is representative of an excavation in which blasting has been poorly controlled. The depth of fracturing could be 0.5m or more. We will model the fracture zone to extend to infinity. In both models the effect of the application of the liner material on the displacement, rotation and tensile stress at the surface of the excavation will be investigated. This could model a thin spray-on liner. Support mechanisms that depend on liner penetration such as the bonding of adjacent blocks which inhibits block movement can be investigated.

The stress and displacement will be calculated using plane strain theory in infinitesimal elasticity. In Section 2 the formulation of plane strain in polar coordinates and in terms of the Airy stress function is outlined. In Section 3 the problem of the stress and displacement around a cylindrical excavation in a shear field is formulated. Two cylindrical regions are considered, an inner region and an outer region which extends to infinity. The normal and tangential components of the displacement and stress are continuous at the interface. In Section 4 a perturbation solution is derived for the stress and displacement to first order in the perturbation parameter which is the ratio of the thickness of the inner region to its radius. In Section 5 the results are applied to the first model of a thin liner and the transfer of stress from the rock to the liner is investigated. In Section 6 the second and third models are considered and the effect on the displacement and stress of the penetration of the liner material into the fractures is investigated. Finally, conclusions are summarised in Section 7.

## 2 Plane strain

We first briefly outline the theory of plane strain and present the equations which will be used later [2, 1].

Let $(r, \theta, z)$ be cylindrical polar coordinates. In plane strain all quantities are independent of $z$ and the displacement in the $z$-direction vanishes:

$$u_r = u_r(r,\theta), \qquad u_\theta = u_\theta(r,\theta), \qquad u_z = 0\,, \tag{2.1}$$

$$e_{rz} = 0\,, \qquad e_{\theta z} = 0\,, \qquad e_{zz} = 0\,, \tag{2.2}$$

$$\tau_{rz} = 0\,, \qquad \tau_{\theta z} = 0\,, \tag{2.3}$$

where $e_{ik}$ is the infinitesimal strain tensor and $\tau_{ik}$ is the stress tensor.

From the inverse Hooke's law,

$$\tau_{zz} = \sigma(\tau_{rr} + \tau_{\theta\theta})\,, \tag{2.4}$$

where $\sigma$ $(0 < \sigma < 1/2)$ is the Poisson ratio. Using (2.4), the remaining components of the inverse Hooke's law that do not vanish identically can be expressed as

$$e_{rr} = \frac{(1-\sigma^2)}{E}\tau_{rr} - \frac{\sigma(1+\sigma)}{E}\tau_{\theta\theta}\,, \tag{2.5}$$

$$e_{r\theta} = \frac{(1+\sigma)}{E}\tau_{r\theta}\,, \tag{2.6}$$

$$e_{\theta\theta} = \frac{(1-\sigma^2)}{E}\tau_{\theta\theta} - \frac{\sigma(1+\sigma)}{E}\tau_{rr}\,, \tag{2.7}$$

where $E$ is the Young's modulus.

The non-zero components of the strain tensor are related to the components, $u_r(r,\theta)$ and $u_\theta(r,\theta)$, of the displacement vector by

$$e_{rr} = \frac{\partial u_r}{\partial r}\,, \tag{2.8}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{2.9}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\,. \tag{2.10}$$

Equations (2.8) to (2.10) consist of three equations for the two displacement components, $u_r$ and $u_\theta$. For a single-valued solution for $u_r$ and $u_\theta$ the components of the strain tensor must satisfy the compatibility condition

$$2\left(\frac{1}{r}\frac{\partial^2 e_{r\theta}}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial e_{r\theta}}{\partial \theta}\right) = \frac{\partial^2 e_{\theta\theta}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 e_{rr}}{\partial \theta^2} + \frac{2}{r}\frac{\partial e_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial e_{rr}}{\partial \theta}\,. \tag{2.11}$$

The full set of six compatibiliy conditions in cylindrical polar coordinates is given by Saada[6]. The remaining five compatibility conditions are identically satisfied because $e_{rz}$, $e_{\theta z}$ and $e_{zz}$ all vanish and there is no dependence on $z$. When expressed in terms of the Cauchy sress tensor using (2.5) to (2.7), the compatibility condition (2.11) becomes

$$\frac{1}{r^2}\frac{\partial^2 \tau_{rr}}{\partial\theta^2} - \frac{\sigma}{1-\sigma}\frac{\partial^2 \tau_{rr}}{\partial r^2} - \frac{2}{1-\sigma}\frac{1}{r}\frac{\partial^2 \tau_{r\theta}}{\partial r\partial\theta} + \frac{\partial^2 \tau_{\theta\theta}}{\partial r^2} - \frac{\sigma}{1-\sigma}\frac{1}{r^2}\frac{\partial^2 \tau_{\theta\theta}}{\partial\theta^2}$$
$$+\left(\frac{2-\sigma}{1-\sigma}\right)\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial r} - \left(\frac{1+\sigma}{1-\sigma}\right)\frac{1}{r}\frac{\partial \tau_{rr}}{\partial r} - \frac{2}{(1-\sigma)}\frac{1}{r^2}\frac{\partial \tau_{r\theta}}{\partial\theta} = 0\,. \tag{2.12}$$

Equation (2.12) is independent of the Young's modulus $E$.

The body force due to gravity will be neglected. The equations of static equilibrium with zero body force are

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}\left(\tau_{rr} - \tau_{\theta\theta}\right) = 0\,, \tag{2.13}$$

$$\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}\tau_{r\theta} + \frac{2}{r}\tau_{r\theta} = 0\,. \tag{2.14}$$

When the body force vanishes, the stress tensor can be expressed in terms of the Airy stress function, $\phi(r,\theta)$, as

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\,, \tag{2.15}$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\,, \tag{2.16}$$

$$\tau_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}\,. \tag{2.17}$$

The equations of static equilibrium, (2.13) and (2.14), are identically satisfied by (2.15) to (2.17).

If (2.15) to (2.17) are substituted into the compatibility condition (2.12), then (2.12) reduces to the biharmonic equation

$$\nabla^4\phi = 0\,, \tag{2.18}$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \tag{2.19}$$

Equation (2.18) is independent of the elastic constants. Hence if the boundary conditions are expressed in terms of tractions the stress in an elastic body in a state of plane strain is independent of the elastic constants.

The rotation vector $\boldsymbol{\Omega}$ of a material element is

$$\boldsymbol{\Omega} = \frac{1}{2}\,\mathrm{curl}\,\boldsymbol{u}\,, \tag{2.20}$$

where $\boldsymbol{u}$ is the displacement vector. For plane strain the displacement vector is given by(2.1). Hence for plane strain

$$\Omega_r = 0\,, \quad \Omega_\theta = 0\,, \quad \Omega_z(r,\theta) = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\theta}{r}\right)\,. \tag{2.21}$$

The amount of strain energy stored per unit volume of an elastic body is sometimes used to determine the limiting stress at which failure occurs [9]. The strain energy is the sum of two parts, that due to change of volume and that due to distortion. Since isotropic elasic materials can sustain very large pressures without yielding only the strain energy due to distortion is considered when determining the strength of the material. By subtracting off the part of the strain energy due to change in volume the part of the strain energy due to distortion is obtained. Expressed in cylindrical polar coordinates the strain energy due to distortion is

$$V_0 = \frac{(1+\sigma)}{6E}\left[(\tau_{rr}-\tau_{\theta\theta})^2+(\tau_{\theta\theta}-\tau_{zz})^2+(\tau_{zz}-\tau_{rr})^2\right] + \frac{(1+\sigma)}{E}\left[\tau_{r\theta}^2+\tau_{\theta z}^2+\tau_{zr}^2\right] . \tag{2.22}$$

For plane strain, using (2.3) and (2.4), $V_0$ reduces to

$$V_0 = \frac{(1+\sigma)}{6E}\left[(\tau_{rr}-\tau_{\theta\theta})^2+\left((1-\sigma)\tau_{\theta\theta}-\sigma\tau_{rr}\right)^2+\left((1-\sigma)\tau_{rr}-\sigma\tau_{\theta\theta}\right)^2\right] + \frac{(1+\sigma)}{E}\tau_{r\theta}^2 . \tag{2.23}$$

# 3 Cylindrical excavation in a shear field

Consider a cylindrical excavation in a uniform shear field $S$ which represents a perturbation to the background stress due to either mining or a geological disturbance caused by stress redistribution induced by slip on a geological structure. Since the elasticity theory is linear the total stress field is the sum of the insitu stress and the stress due to the perturbation. We will be concerned here only with the stress due to the perturbation.

Consider a cylindrical excavation of radius $a$ as shown in Figure 1. The elastic medium $a \le r \le \infty$ consists of region 1 ($a \le r \le b$) and region 2 ($b \le r \le \infty$). Quantities in region 1 are denoted by a subscript or superscript 1 and in region 2 by a subscript or superscript 2. Using the solution for the stress and displacement for this problem the three models can be analysed.

At large distances from the excavation,

$$r \to \infty : \qquad \tau_{xx}^{(2)} = 0\,, \quad \tau_{xy}^{(2)} = S\,, \quad \tau_{yy}^{(2)} = 0\,, \tag{3.1}$$

where $S$ is a constant. But in terms of Cartesian coordinates [2],

$$\tau_{xx} = \frac{\partial^2\phi}{\partial y^2}\,, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}\,, \quad \tau_{yy} = \frac{\partial^2\phi}{\partial x^2}\,. \tag{3.2}$$

Substituting (3.2) into (3.1) and solving for the Airy stress function gives

$$r \to \infty : \qquad \phi_2 = -Sxy = -\tfrac{1}{2}Sr^2\sin 2\theta\,. \tag{3.3}$$

The additive constants in (3.3) have been dropped because they do not contribute to the stress.

The Airy stress functions $\phi_1(r,\theta)$ and $\phi_2(r,\theta)$ satisfy the biharmonic equaion:

$$a \le r \le b : \qquad \nabla^4\phi_1 = 0\,, \tag{3.4}$$

$$b \le r \le \infty : \qquad \nabla^4\phi_2 = 0\,. \tag{3.5}$$

The surface $r = a$ is traction-free. The adhesive liner is firmly bonded to the rock at the interface $r = b$. Hence in all three models the normal and tangential components of the displacement and stress are continuous at the interface. The boundary conditions are:

$$r = a : \qquad \tau_{rr}^{(1)}(a,\theta) = 0\,, \tag{3.6}$$

$$r = a : \qquad \tau_{r\theta}^{(1)}(a,\theta) = 0\,, \tag{3.7}$$

$$r = b : \qquad \tau_{rr}^{(1)}(b,\theta) = \tau_{rr}^{(2)}(b,\theta)\,, \tag{3.8}$$

$$r = b : \qquad \tau_{r\theta}^{(1)}(b,\theta) = \tau_{r\theta}^{(2)}(b,\theta)\,, \tag{3.9}$$

$$r = b : \qquad u_r^{(1)}(b,\theta) = u_r^{(2)}(b,\theta)\,, \tag{3.10}$$

$$r = b : \qquad u_\theta^{(1)}(b,\theta) = u_\theta^{(2)}(b,\theta)\,, \tag{3.11}$$

$$r \to \infty : \qquad \phi_2(r,\theta) \to -\tfrac{1}{2}\,Sr^2\sin 2\theta\,. \tag{3.12}$$

The boundary conditions (3.10) and (3.11) depend on the elastic constants in regions 1 and 2. Unlike the problem of a cylindrical hole in an infinite elastic medium this problem cannot be formulated in terms of stress alone. The solution will depend on the elastic constants in regions 1 and 2.

Because of the form of the boundary condition (3.12) we look for a solution of the form [2]

$$\phi(r,\theta) = f(r)\sin 2\theta\,. \tag{3.13}$$

Equation (3.13) is a solution of the biharmonic equation provided

$$\frac{d^4 f}{dr^4} + \frac{2}{r}\frac{d^3 f}{dr^3} - \frac{9}{r^2}\frac{d^2 f}{dr^2} + \frac{9}{r^3}\frac{df}{dr} = 0 . \tag{3.14}$$

Since (3.14) is an equidimensional equation in $r$ we look for a solution of the form

$$f(r) = kr^n , \tag{3.15}$$

where $k$ is a constant. It can be verified that (3.14) is satisfied provided $n = 4, 2, 0, -2$ and therefore

$$\phi_1(r,\theta) = \left[Ar^4 + Br^2 + C + \frac{D}{r^2}\right] \sin 2\theta , \tag{3.16}$$

$$\phi_2(r,\theta) = \left[Mr^4 + Nr^2 + P + \frac{Q}{r^2}\right] \sin 2\theta , \tag{3.17}$$

where $A$, $B$, $C$, $D$, $M$, $N$, $P$ and $Q$ are constants. From the boundary condition (3.12),

$$M = 0 , \qquad N = -\frac{S}{2} . \tag{3.18}$$

The components of the Cauchy stress tensor in region 1 are, from (2.15) to (2.17),

$$\tau_{rr}^{(1)}(r,\theta) = \left[-2B - \frac{4C}{r^2} - \frac{6D}{r^4}\right] \sin 2\theta , \tag{3.19}$$

$$\tau_{r\theta}^{(1)}(r,\theta) = \left[-6Ar^2 - 2B + \frac{2C}{r^2} + \frac{6D}{r^4}\right] \cos 2\theta , \tag{3.20}$$

$$\tau_{\theta\theta}^{(1)}(r,\theta) = \left[12Ar^2 + 2B + \frac{6D}{r^4}\right] \sin 2\theta . \tag{3.21}$$

On setting $A = 0$, $B = -S/2$, $C = P$ and $D = Q$ in (3.19) to (3.21), the components of the stress tensor in region 2 are obtained:

$$\tau_{rr}^{(2)}(r,\theta) = \left[S - \frac{4P}{r^2} - \frac{6Q}{r^4}\right] \sin 2\theta , \tag{3.22}$$

$$\tau_{r\theta}^{(2)}(r,\theta) = \left[S + \frac{2P}{r^2} + \frac{6Q}{r^4}\right] \cos 2\theta , \tag{3.23}$$

$$\tau_{\theta\theta}^{(2)}(r,\theta) = \left[-S + \frac{6Q}{r^4}\right] \sin 2\theta \tag{3.24}$$

The four boundary conditions on the stress, (3.6) to (3.9), becomes

$$a^4 B + 2a^2 C + 3D = 0 , \tag{3.25}$$

$$3a^6 A + a^4 B - a^2 C - 3D = 0 , \tag{3.26}$$

$$b^4 B + 2b^2 C + 3D - 2b^2 P - 3Q = -\tfrac{1}{2} b^4 S , \tag{3.27}$$

$$3b^6 A + b^4 B - b^2 C - 3D + b^2 P + 3Q = -\tfrac{1}{2} b^4 S . \tag{3.28}$$

It remains to impose the displacement boundary conditions (3.10) and (3.11). Consider first $u_r(r,\theta)$ and $u_\theta(r,\theta)$ in region 1. Substitute (3.19) to (3.21) into the inverse Hooke's law (2.5) to (2.7) and use (2.8) to (2.10) for the strain tensor in terms of the displacement. This leads to the following three first order partial differential equations for $u_r(r,\theta)$ and $u_\theta(r,\theta)$:

$$\frac{\partial u_r}{\partial r} = \frac{(1+\sigma)}{E}\left[-12\sigma A r^2 - 2B - 4(1-\sigma)\frac{C}{r^2} - \frac{6D}{r^4}\right]\sin 2\theta , \tag{3.29}$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{(1+\sigma)}{E}\left[-12Ar^2 - 4B + \frac{4C}{r^2} + \frac{12D}{r^4}\right]\cos 2\theta , \tag{3.30}$$

$$\frac{\partial u_\theta}{\partial \theta} + u_r = \frac{(1+\sigma)}{E}\left[12(1-\sigma)Ar^3 + 2Br + 4\sigma\frac{C}{r} + \frac{6D}{r^3}\right]\sin 2\theta . \tag{3.31}$$

The three equations, (3.29) to (3.31), for the two unknowns, $u_r$ and $u_\theta$, are compatible because the compatibility condition (2.11) is satisfied. Equations (3.29) and (3.31) are integrated with respect to $r$ and $\theta$, respectively, to give $u_r(r,\theta)$ and $u_\theta(r,\theta)$. The arbitrary functions of integration are obtained by substituting $u_r$ and $u_\theta$ into (3.30) and using the technique of separation of variables. This gives

$$u_r^{(1)}(r,\theta) = \frac{(1+\sigma_1)}{E_1}\left[-4\sigma_1 A r^3 - 2Br + 4(1-\sigma_1)\frac{C}{r} + \frac{2D}{r^3}\right]\sin 2\theta$$
$$-F_1 \sin\theta + G_1 \cos\theta , \tag{3.32}$$

$$u_\theta^{(1)}(r,\theta) = \frac{(1+\sigma_1)}{E_1}\left[-2(3-2\sigma_1)Ar^3 - 2Br + 2(1-2\sigma_1)\frac{C}{r} - \frac{2D}{r^3}\right]\cos 2\theta$$

$$-F_1 \cos\theta - G_1 \sin\theta + H_1 r \ , \tag{3.33}$$

where $F_1$, $G_1$ and $H_1$ are constants. The displacement in region 2 is obtained by setting $A = 0$, $B = -S/2$, $C = P$ and $D = Q$ in (3.32) and (3.33):

$$u_r^{(2)}(r,\theta) = \frac{(1+\sigma_2)}{E_2}\left[Sr + 4(1-\sigma_2)\frac{P}{r} + \frac{2Q}{r^3}\right]\sin 2\theta$$

$$-F_2 \sin\theta + G_2 \cos\theta \ , \tag{3.34}$$

$$u_\theta^{(2)}(r,\theta) = \frac{(1+\sigma_2)}{E_2}\left[Sr + 2(1-2\sigma_2)\frac{P}{r} - \frac{2Q}{r^3}\right]\cos 2\theta$$

$$-F_2 \cos\theta - G_2 \sin\theta + H_2 r \ , \tag{3.35}$$

where $F_2$, $G_2$ and $H_2$ are constants.

Equations (3.32) to (3.35) are substituted into the boundary conditions (3.10) and (3.11). This yields

$$F_1 = F_2 = F \ , \qquad G_1 = G_2 = G \ , \qquad H_1 = H_2 = H \ , \tag{3.36}$$

and

$$2\sigma_1 b^6 A + b^4 B - 2(1-\sigma_1)b^2 C - D + 2(1-\sigma_2)\alpha b^2 P + \alpha Q = -\tfrac{1}{2}\,\alpha b^4 S \ , \tag{3.37}$$

$$(3-2\sigma_1)b^6 A + b^4 B - (1-2\sigma_1)b^2 C + D + (1-2\sigma_2)\alpha b^2 P - \alpha Q = -\tfrac{1}{2}\,\alpha b^4 S \ , \tag{3.38}$$

where

$$\alpha = \left(\frac{1+\sigma_2}{1+\sigma_1}\right)\frac{E_1}{E_2} = \left(\frac{1-\sigma_1}{1-\sigma_2}\right)\frac{E_1'}{E_2'} \ , \qquad E' = \frac{E}{1-\sigma^2} \ . \tag{3.39}$$

We find that it is the ratio $E_1'/E_2'$ that occurs naturally in the final solution.

The displacement

$$u_r(r,\theta) = -F\sin\theta + G\cos\theta \ , \qquad u_\theta(r,\theta) = -F\cos\theta - G\sin\theta + H \ , \tag{3.40}$$

when expressed in Cartesian coordinates is

$$u_x(x,y) = G - Hy \ , \qquad u_y(x,y) = -F + Hx \ . \tag{3.41}$$

Equation (3.41) describes a rigid body translation of $G$ and $-F$ in the $x$ and $y$ directions and a rigid body rotation $H$ about the $z$ axis. We therefore take $G = F = H = 0$.

The boundary conditions (3.25) to (3.28), (3.37) and (3.38) are six equations for the six unknown constants, $A$, $B$, $C$, $D$, $P$ and $Q$. A perturbation solution for the six constants will be derived in the next section.

# 4 Perturbation solution

Let

$$\varepsilon = \frac{b-a}{a} \tag{4.1}$$

Since the thickness of the liner and the distance of penetration of the liner material into the rock fractures are small compared with the radius of the excavation, $\varepsilon$ will be small. For an excavation of radius $a = 2$m and a liner of thickness $b - a = 50$mm, $\varepsilon = 2.5 \times 10^{-2}$ while if the depth of penetration $b - a = 200$mm then $\varepsilon = 0.1$. We will derive a perturbation solution for the stress, displacement and rotation in regions 1 and 2 correct to first order in the perturbation parameter $\varepsilon$.

Now

$$b = a(1+\varepsilon) . \tag{4.2}$$

We will derive the perturbation solution in terms of $a$. It can readily be expressed in terms of $b$ using

$$a = b\big(1 - \varepsilon + \mathrm{O}(\varepsilon^2)\big) , \tag{4.3}$$

as $\varepsilon \to 0$. Let

$$\begin{aligned} A &= A_0 + \varepsilon A_1 + \mathrm{O}(\varepsilon^2) , & D &= D_0 + \varepsilon D_1 + \mathrm{O}(\varepsilon^2) , \\ B &= B_0 + \varepsilon B_1 + \mathrm{O}(\varepsilon^2) , & P &= P_0 + \varepsilon P_1 + \mathrm{O}(\varepsilon^2) , \\ C &= C_0 + \varepsilon C_1 + \mathrm{O}(\varepsilon^2) , & Q &= Q_0 + \varepsilon Q_1 + \mathrm{O}(\varepsilon^2) , \end{aligned} \tag{4.4}$$

as $\varepsilon \to 0$. Equations (4.2) and (4.4) are substituted into the boundary conditions (3.25) to (3.28), (3.37) and (3.38). The boundary conditions are then expanded in powers of $\varepsilon$.

## 4.1 Zero order in $\varepsilon$

To zero order in $\varepsilon$ the boundary conditions are

$$a^4 B_0 + 2a^2 C_0 + 3D_0 = 0 , \tag{4.5}$$

$$3a^6 A_0 + a^4 B_0 - a^2 C_0 - 3D_0 = 0 , \tag{4.6}$$

$$2a^2 P_0 + 3Q_0 = \tfrac{1}{2} a^4 S , \tag{4.7}$$

$$a^2P_0 + 3Q_0 = -\tfrac{1}{2}\,a^4S\,, \tag{4.8}$$

$$2\sigma_1 a^6A_0 + a^4B_0 - 2(1-\sigma_1)a^2C_0 - D_0 + 2(1-\sigma_2)\alpha a^2P_0 + \alpha Q_0 = -\tfrac{1}{2}\,\alpha a^4S\,, \tag{4.9}$$

$$(3-2\sigma_1)a^6A_0 + a^4B_0 - (1-2\sigma_1)a^2C_0 + D_0 + (1-2\sigma_2)\alpha a^2P_0 - \alpha Q_0 = -\tfrac{1}{2}\,\alpha a^4S\,. \tag{4.10}$$

Equation (4.5) was used to eliminate $B_0$, $C_0$ and $D_0$ from (4.7) while (4.6) was used to eliminate $A_0$, $B_0$, $C_0$ and $D_0$ from (4.8). Equations (4.7) and (4.8) are readily solved for $P_0$ and $Q_0$. From (4.5) and (4.6),

$$C_0 = -(3a^4A_0 + 2a^2B_0)\,, \qquad D_0 = 2a^6A_0 + a^4B_0 \tag{4.11}$$

and substituting (4.11) into (4.9) and (4.10) gives two equations for $A_0$ and $B_0$. It can be verified that

$$\begin{aligned} &A_0 = 0\,, && B_0 = -\frac{1}{2}\frac{E_1'}{E_2'}\,S\,, && C_0 = \frac{E_1'}{E_2'}\,a^2S\,, \\ &D_0 = -\frac{1}{2}\frac{E_1'}{E_2'}\,a^4S\,, && P_0 = a^2S\,, && Q_0 = -\frac{1}{2}\,a^4S\,. \end{aligned} \tag{4.12}$$

## 4.2 First order in $\varepsilon$

Equating the coefficients of $\varepsilon$ in the boundary conditions gives

$$a^4B_1 + 2a^2C_1 + 3D_1 = 0\,, \tag{4.13}$$

$$3a^6A_1 + a^4B_1 - a^2C_1 - 3D_1 = 0\,, \tag{4.14}$$

$$2a^2P_1 + 3Q_1 = 2a^4S + 4a^4B_0 + 4a^2C_0 - 4a^2P_0\,, \tag{4.15}$$

$$a^2P_1 + 3Q_1 = -2a^4S - 18a^6A_0 - 4a^4B_0 + 2a^2C_0 - 2a^2P_0\,, \tag{4.16}$$

$$\begin{aligned} &2\sigma_1 a^6A_1 + a^4B_1 - 2(1-\sigma_1)a^2C_1 - D_1 + 2(1-\sigma_2)\alpha a^2P_1 + \alpha Q_1 \\ &= -2\alpha a^4S - 12\sigma_1 a^6A_0 - 4a^4B_0 + 4(1-\sigma_1)a^2C_0 - 4(1-\sigma_2)\alpha a^2P_0\,, \end{aligned} \tag{4.17}$$

$$\begin{aligned} &(3-2\sigma_1)a^6A_1 + a^4B_1 - (1-2\sigma_1)a^2C_1 + D_1 + (1-2\sigma_2)\alpha a^2P_1 - \alpha Q_1 \\ &= -2\alpha a^4S - 6(3-2\sigma_1)a^6A_0 - 4a^4B_0 + 2(1-2\sigma_1)a^2C_0 - 2(1-2\sigma_2)\alpha a^2P_0\,. \end{aligned} \tag{4.18}$$

The constants $B_1$, $C_1$ and $D_1$ were eliminated from (4.15) using (4.13). The constants $A_1$, $B_1$, $C_1$ and $D_1$ were eliminated from (4.16) using (4.14). The structure of the left hand side of the first order perturbation equations, (4.13) to (4.18), is the same as that of the zero order perturbation equations, (4.5) to (4.10). The first order system is therefore solved in the same way as the zero order system. It is found that

$$A_1 = -\frac{1}{2}\left[\frac{1}{1-\sigma_1} - \frac{1}{(1-\sigma_2)}\frac{E_1'}{E_2'}\right]\frac{E_1'}{E_2'}\frac{S}{a^2}\,, \tag{4.19}$$

$$B_1 = \left[\frac{\sigma_1}{1-\sigma_1} - \frac{\sigma_2}{(1-\sigma_2)}\frac{E_1'}{E_2'}\right]\frac{E_1'}{E_2'}S\,, \tag{4.20}$$

$$C_1 = \frac{1}{2}\left[\frac{3-4\sigma_1}{1-\sigma_1} - \left(\frac{3-4\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'}\right]\frac{E_1'}{E_2'}a^2 S\,, \tag{4.21}$$

$$D_1 = -\left[1 - \frac{E_1'}{E_2'}\right]\frac{E_1'}{E_2'}a^4 S\,, \tag{4.22}$$

$$P_1 = 2\left[1 - \frac{E_1'}{E_2'}\right]a^2 S\,, \tag{4.23}$$

$$Q_1 = -2\left[1 - \frac{E_1'}{E_2'}\right]a^4 S\,. \tag{4.24}$$

## 4.3 Solutions correct to first order in $\varepsilon$

The solutions for the stress, displacement, rotation and strain energy due to distortion can now be obtained correct to first order in $\varepsilon$.

Consider first region 1 at the surface $r = a$. Correct to first order in $\varepsilon$,

$$\tau_{\theta\theta}^{(1)}(a,\theta) = -4\frac{E_1'}{E_2'}S\left[1 + \left(\frac{3-2\sigma_1}{1-\sigma_1} - \left(\frac{3-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'}\right)\varepsilon\right]\sin 2\theta\,, \tag{4.25}$$

$$u_r^{(1)}(a,\theta) = \frac{4aS}{E_2'}\left[1 + \left(\frac{1-2\sigma_1}{1-\sigma_1} - \left(\frac{1-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'}\right)\varepsilon\right]\sin 2\theta\,, \tag{4.26}$$

$$u_\theta^{(1)}(a,\theta) = \frac{4aS}{E_2'}\left[1 + 2\left(1 - \frac{E_1'}{E_2'}\right)\varepsilon\right]\cos 2\theta\,, \tag{4.27}$$

$$\Omega_z^{(1)}(a,\theta) = -\frac{4S}{E_2'}\left[1 + 2\left(\frac{\sigma_2}{(1-\sigma_2)}\frac{E_1'}{E_2'} - \frac{\sigma_1}{1-\sigma_1}\right)\varepsilon\right]\cos 2\theta\,, \tag{4.28}$$

$$V_0^{(1)}(a,\theta) = \frac{8}{3}\frac{\left[1+(1-\sigma_1)^2+\sigma_1^2\right]}{(1-\sigma_1)}\frac{E_1'}{E_2'^2}S^2\left[1+\right.$$

$$\left.+2\left(\frac{3-2\sigma_1}{1-\sigma_1}-\left(\frac{3-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'}\right)\varepsilon\right]\sin^2 2\theta\,. \quad (4.29)$$

Consider next region 2 at the interface $r = b$. We express $a$ in terms of $b$ using (4.3). Then, correct to first order in $\varepsilon$,

$$\tau_{rr}^{(2)}(b,\theta) = -4\varepsilon\frac{E_1'}{E_2'}S\sin 2\theta\,, \quad (4.30)$$

$$\tau_{r\theta}^{(2)}(b,\theta) = 8\varepsilon\frac{E_1'}{E_2'}S\cos 2\theta\,, \quad (4.31)$$

$$\tau_{\theta\theta}^{(2)}(b,\theta) = -4S\left[1-3\frac{E_1'}{E_2'}\varepsilon\right]\sin 2\theta\,, \quad (4.32)$$

$$u_r^{(2)}(b,\theta) = \frac{4bS}{E_2'}\left[1-\left(\frac{1-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'}\varepsilon\right]\sin 2\theta\,, \quad (4.33)$$

$$u_\theta^{(2)}(b,\theta) = \frac{4bS}{E_2'}\left[1-2\frac{E_1'}{E_2'}\varepsilon\right]\cos 2\theta\,, \quad (4.34)$$

$$\Omega_z^{(2)}(b,\theta) = -\frac{4S}{E_2'}\left[1-2\frac{E_1'}{E_2'}\varepsilon\right]\cos 2\theta\,, \quad (4.35)$$

$$V_0^{(2)}(b,\theta) = \frac{8}{3}\frac{(1+\sigma_2)}{E_2}\left[1+(1-\sigma_2)^2+\sigma_2^2\right]S^2\left[1\right.$$

$$\left.-2\frac{(5+2(1-2\sigma_2)+4\sigma_2^2)}{(1+(1-\sigma_2)^2+\sigma_2^2)}\frac{E_1'}{E_2'}\varepsilon\right]\sin^2 2\theta\,. \quad (4.36)$$

The results correct to first order in $\varepsilon$ are analysed in Sections 5 and 6. The conclusions assume that terms of order $\varepsilon^2$ can be neglected.

# 5 Thin adhesive liner without joint and fracture penetration

In this section we consider the first model in which a thin liner forms a cylindrical layer firmly bonded to the inner surface of the excavation. There is no penetration of the liner material into joints and fractures in the rock mass. The liner is thin if its thickness is less than 20mm. Region 1, $a \le r \le b$, is the liner and region 2, $b \le r \le \infty$, is the rock. We compare the stress, strain energy due to distortion, displacement and rotation in region 2 at the interface $r = b$ with corresponding values in a cylindrical excavation of radius $b$ without a liner.

Consider first the stress concentration factor which can be defined in terms of either the maximum shear stress or the maximum tensile stress [2]. Since the excavation is subjected to shear consider first the shear stress. Then

$$\textit{stress concentration factor} = \frac{\textit{maximum shear stress due to excavation}}{\textit{maximum shear stress in absence of excavation}} .$$

Now, at a given point, the maximum shear stress equals one-half of the magnitude of the difference between the greatest and least principal stresses at that point [5]. Since $\tau_{r\theta} = O(\varepsilon)$, the principal stresses correct to order $\varepsilon$ are $\tau_{rr}$, $\tau_{\theta\theta}$, $\tau_{zz}$ where $\tau_{zz}$ is given by (2.4). A suitable point to consider is $r = b$, $\theta = 3\pi/4$. In the absence of a liner, this point is on the surface of the excavation and $\tau_{\theta\theta}$ attains its maximum value of $4S$ and from the boundary condition $\tau_{rr} = 0$. In the presence of a liner the greatest and least principal stress at this point are $\tau^{(2)}_{\theta\theta}(b, 3\pi/4)$ and $\tau^{(2)}_{rr}(b, 3\pi/4)$. Hence, using (4.32) and (4.30),

$$\textit{maximum shear stress at } \left(b, \frac{3\pi}{4}\right) = 2S\left(1 - 4\frac{E_1'}{E_2'}\varepsilon\right) . \qquad (5.1)$$

Since the shear stress in the absence of the excavation is $S$ it follows that

$$\textit{stress concentration factor (shear stress)} = 2\left(1 - 4\frac{E_1'}{E_2'}\varepsilon\right) . \qquad (5.2)$$

The stress concentration factor can also be defined as:

$$\textit{stress concentration factor} = \frac{\textit{maximum tensile stress due to excavation}}{\textit{maximum tensile stress in absence of excavation}} .$$

This definition may be appropriate for a brittle material [2]. The greatest normal stress acting on any plane through a given point is the greatest principal stress at that point [5]. Consider again the point $r = b$, $\theta = 3\pi/4$. Then

$$\textit{maximum tensile stress} = \tau_{\theta\theta}^{(2)}\left(b, \frac{3\pi}{4}\right) = 4S\left(1 - 3\frac{E_1'}{E_2'}\varepsilon\right). \qquad (5.3)$$

When there is no excavation the principal stresses are $S$, 0, $-S$ and therefore the maximum tensile stress is $S$. Thus

$$\textit{stress concentration factor (tensile stress)} = 4\left(1 - 3\frac{E_1'}{E_2'}\varepsilon\right). \qquad (5.4)$$

The stress concentration factor is a measure of the severity of the stress field. When there is no liner the stress concentration factor is obtained by setting $\varepsilon = 0$. The stress concentration factors, (5.2) and (5.4), at the interface are both reduced by the presence of the liner. Stress is transferred from the rock to the liner. The greater the ratio $E_1'/E_2'$ the more stress will be transferred. The stiffer liner will attract more stress from the rock. It will therefore be better in preventing facture and in inhibiting the onset of the movement of rock. However, it is known that once rock movement has started a flexible liner will be better for support than a stiff liner [8]. The decrease in the stress concentration factor is proportional to $\varepsilon$ and therefore linearly proportional to the thickness of the liner. The thicker the liner the greater the transfer of stress from the rock to the liner.

When $E_1' = E_2'$ the stress concentration factor does not depend on the elastic constants. The liner acts as an arch. If $E_1' > E_2'$ the support given by the liner exceeds the arching action while if $E_1' < E_2'$ it is less than the arching action.

The strain energy due to distortion in the rock at the interface $r = b$, given by (4.36), is clearly decreased by the presence of the liner. The decrease is proportional to the ratio $E_1'/E_2'$ and is greater for a stiff liner. The decrease is also proportional to $\varepsilon$ and hence is linearly proportional to the thickness of the liner. Since the strain energy due to distortion can be used to determine the limiting stress at which failure occurs, the effect of the liner is to inhibit the onset of failure in the rock. The rock with an adhesive liner can sustain a larger shear perturbation before failure occurs than rock without an adhesive liner.

The magnitude of the radial and tangential components of the displacement at the interface, $u_r^{(2)}(b,\theta)$ and $u_\theta^{(2)}(b,\theta)$, given by (4.33) and (4.34), are both decreased due to the presence of the adhesive liner. Again the decrease is proportional to $E_1'/E_2'$ and $\varepsilon$. The decrease is greatest for a stiff liner and is proportional to the thickness of the liner. Since $u_r^{(2)}(b,\theta)$ is proportional to $\sin 2\theta$ and $u_\theta^{(2)}(b,\theta)$ is proportional to $\cos 2\theta$, the radial displacement vanishes when the magnitude of the tangential displacement is a maximum and conversely the tangential displacement vanishes when the magnitude of the radial displacement is a maximum. The deformation of the rock due to the shear perturbation is inhibited by the adhesive liner.

The rotation of the rock at the interface, $\Omega_z^{(2)}(b,\theta)$, is given by (4.35). We see that the bonding of the liner to the rock restricts the rotation of the rock. The reduction in the rotation is proportional to $E_1'/E_2'$ and $\varepsilon$. Hence stiffer and thicker adhesive liners will be more effective in reducing block rotation. The rotation $\Omega_z^{(2)}(b,\theta)$ is proportional to $\cos 2\theta$ while the hoop stress $\tau_{\theta\theta}^{(2)}(b,\theta)$ is proportional to $\sin 2\theta$. At points on the interface where the hoop stress in the rock vanishes the magnitude of the rotation in the rock is a maximum and at points where the rotation in the rock vanishes the magnitude of the hoop stress is a maximum.

The tensile stress in the liner at $r = a$ is the hoop stress $\tau_{\theta\theta}^{(1)}(a,\theta)$ given by (4.25). Unlike the hoop stress in the rock at $r = b$ given by (4.32), the tensile stress in the liner is proportional to the ratio $E_1'/E_2'$. The stiffer liner therefore attracts stress [8].

# 6 Joint and fracture penetration by liner material

In this section we consider the two models in which the surface of the excavation is $r = a$ and the liner material penetrates into the open joints and fractures. The thin layer of liner at the surface will be neglected since its thickness is typically 4mm or less. The effect of the liner will be modelled only by the penetration of the liner material into the joints and fractures in the rock.

The effect of cracks on the elastic properties of rock has been reviewed by Jaeger and Cook [3]. The effective Young's modulus of a body containing open cracks and cavities and of a body containing closed cavities if the surfaces of the cracks slide past one another is less than the intrinsic Young's modulus of the solid body. The effective Poisson ratio of a body containing equidimensional cavities or containing very flat open cracks is less than the intrinsic Poisson ratio of the solid body. (The effective Poisson ratio of a body containing closed cracks is greater than its intrinsic Poisson ratio.)

## 6.1 Penetration of liner material into fractures to their full extent

We now consider the second model in which the liner material penetrates into the fractures to their full extent. The rock outside the liner penetration contains no fractures. We will compare the displacement and stress at the surface of the excavation, $r = a$, before and after the application of the liner material.

Region 1, $a \leq r \leq b$, consists of rock with open joints and fractures and region 2, $b \leq r \leq \infty$, consists of rock without fractures. Before the application of the liner material, $E_1 < E_2$ and $\sigma_1 < \sigma_2$. We will assume that after the open joints and fractures have been penetrated by the liner material the Young's modulus and Poisson ratio of the rock with the liner material is the same as the intrinsic Young's modulus and Poisson ratio of the rock in region 2. Thus after the liner material has been applied $E_1 = E_2$ and $\sigma_1 = \sigma_2$ and the stress and displacement will be independent of $\varepsilon$.

Consider first the radial and tangential displacement at the surface of the excavation due to the shear perturbation. If the perturbation occurs before the application of the liner material then $u_r^{(1)}(a, \theta)$ and $u_\theta^{(1)}(a, \theta)$ are given by (4.26) and (4.27). If the perturbation occurs after the liner material is applied then, setting $\sigma_1 = \sigma_2$ and $E_1' = E_2'$ in (4.26) and (4.27),

$$u_r^{(1)}(a, \theta) = \frac{4aS}{E_2'} \sin 2\theta \,, \qquad u_\theta^{(1)}(a, \theta) = \frac{4aS}{E_2'} \cos 2\theta \,. \tag{6.1}$$

Now $\sigma_1 < \sigma_2$, $E_1 < E_2$ and therefore $E_1' < E_2'$. Also to investigate the magnitude of (4.26) we observe that

$$\frac{1-2\sigma_1}{1-\sigma_1} - \left(\frac{1-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'} = \left(\frac{1-2\sigma_2}{1-\sigma_2}\right)\left[\left(\frac{1-2\sigma_1}{1-2\sigma_2}\right)\left(\frac{1-\sigma_2}{1-\sigma_1}\right) - \frac{E_1'}{E_2'}\right] > 0 \tag{6.2}$$

since

$$\left(\frac{1-2\sigma_1}{1-2\sigma_2}\right)\left(\frac{1-\sigma_2}{1-\sigma_1}\right) > 1 \,. \tag{6.3}$$

Hence the magnitudes of the displacements in (6.1) are less than the magnitudes of (4.26) and (4.27). The reduction in magnitude is proportional to $a\varepsilon$, the depth of penetration of the liner material. Hence the penetration of the liner material into the joints and fractures reduces the magnitude of the radial and tangential displacement at the surface of the excavation and this reduction increases linearly with the depth of penetration.

Consider now the rotation of a material element at the surface of the excavation due to the shear perturbation. If the perturbation occurs before the liner material is applied then $\Omega_z^{(1)}(a,\theta)$ is given by (4.28). If it occurs after the application of the liner material then, setting $\sigma_1 = \sigma_2$ and $E_1' = E_2'$ in (4.28),

$$\Omega_z^{(1)}(a,\theta) = -4\frac{S}{E_2'}\cos 2\theta \, . \tag{6.4}$$

Now to investigate the magnitude of (4.28) we observe that

$$\frac{\sigma_2}{(1-\sigma_2)}\frac{E_1'}{E_2'} - \frac{\sigma_1}{1-\sigma_1} = \frac{\sigma_2(1+\sigma_2)}{(1-\sigma_1^2)}\left[\frac{E_1}{E_2} - \frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)}\right] . \tag{6.5}$$

But since $\sigma_1 < \sigma_2$,

$$\frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} < 1 \tag{6.6}$$

and we also have

$$\frac{E_1}{E_2} < 1 \, . \tag{6.7}$$

Thus if

$$\frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} < \frac{E_1}{E_2} < 1 \, , \tag{6.8}$$

then the magnitude of the rotation is decreased and penetration of the liner material into joints and cracks will inhibit rotational movement of blocks [8]. However if

$$\frac{E_1}{E_2} < \frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} < 1 \, , \tag{6.8}$$

then the magnitude of the rotation is increased. This case applies if the Poisson ratio of the rock is not altered significantly by the cracks. Since the displacement is decreased by the penetration of the liner material into joints and cracks it is not unreasonable that the rotation could increase because the work done by the shear perturbation will be the same with or without the application of liner material. The magnitude of the change in rotation is proportional to $\varepsilon$ and hence it increases linearly with the depth of penetration of the liner material.

Consider next the hoop stress $\tau_{\theta\theta}^{(1)}(a,\theta)$ at the surface of the excavation. If the shear perturbation occurs before the liner material is applied then the hoop stress is given by (4.25). If it occurs after the liner material is applied then, setting $\sigma_1 = \sigma_2$ and $E_1' = E_2'$ in (4.25),

$$\tau_{\theta\theta}^{(1)}(a,\theta) = -4S\sin 2\theta \, . \tag{6.10}$$

Unlike the displacement and rotation given by (4.26), (4.27) and (4.28), the hoop stress (4.25) is proportional to $E_1'/E_2'$. The change in magnitude due to the application of the liner material is therefore more significant for the hoop stress than for the displacement and rotation. The magnitude of the hoop stress is increased by a factor $E_2'/E_1'$ if the shear perturbation occurs after the application of the liner material. The penetration of the liner material into joints and cracks will bond blocks together and inhibit movement of blocks. This extra bonding results in a greater hoop stress when the shear perturbation takes place.

Finally, consider the strain energy due to distortion at the surface of the excavation, $V_0^{(1)}(a,\theta)$. It depends only on $\tau_{\theta\theta}^{(1)}(a,\theta)$ because $\tau_{rr}^{(1)}(a,\theta)$ and $\tau_{r\theta}^{(1)}(a,\theta)$ both vanish from the boundary conditions (3.6) and (3.7). If the shear perturbation occurs before the application of the liner material then $V_0^{(1)}(a,\theta)$ is given by (4.29). We see that $V_0^{(1)}(a,\theta)$ is proportional to $E_1'/E_2'^2$. If the shear perturbation occurs after the liner material has been applied then setting $\sigma_1 = \sigma_2$ and $E_1' = E_2'$ in (4.29),

$$V_0^{(1)}(a,\theta) = \frac{8}{3}\frac{\left[1+(1-\sigma_2)^2+\sigma_2^2\right]}{(1-\sigma_2)E_2'}\,S^2\sin^2 2\theta\,. \qquad (6.11)$$

The strain energy due to distortion is greater by a factor $E_2'/E_1'$ if the shear perturbation occurs after the liner material has been applied. The bonding of blocks together due to the penetration of liner material into joints and cracks prevents the loosening of the rock mass due to the shear perturbation. The strain energy due to distortion is increased as a result.

## 6.2 Fracture zone large compared with penetration distance

Finally, we consider the third model in which the liner material penetrates some distance into the fractures but the fractures extend well beyond this depth. We will make the simplification that the fracture zone extends to infinity.

Before the application of the liner material, region 1 ($a \le r \le b$) and region 2 ($b \le r \le \infty$) consist of fractured rock with $\sigma_1 = \sigma_2$, $E_1 = E_2$ and $E_1' = E_2'$. The stress and displacement will be independent of $\varepsilon$. After the application of the liner material, region 1 is penetrated to its full extent and region 2 is not penetrated by liner material. Hence $\sigma_1 > \sigma_2$, $E_1 > E_2$ and $E_1' > E_2'$.

Consider the radial and tangential components of the displacement at the surface of the excavation. If the shear perturbation occurs before the application of the liner material then $\sigma_1 = \sigma_2$, $E_1' = E_2'$ and $u_r^{(1)}(a,\theta)$ and $u_\theta^{(1)}(a,\theta)$ are given by (6.1). If the perturbation occurs after the application of the liner material then $u_r^{(1)}(a,\theta)$ and $u_\theta^{(1)}(a,\theta)$ are given by (4.26) and (4.27) with $\sigma_1 > \sigma_2$ and $E_1' > E_2'$. Clearly the magnitude of $u_\theta^{(1)}(a,\theta)$ is decreased by the application of liner material. The magnitude of $u_r^{(1)}(a,\theta)$ is also decreased because

$$\frac{1-2\sigma_1}{1-\sigma_1} - \left(\frac{1-2\sigma_2}{1-\sigma_2}\right)\frac{E_1'}{E_2'} = -\left(\frac{1-2\sigma_1}{1-2\sigma_2}\right)\left[\frac{E_1'}{E_2'} - \left(\frac{1-2\sigma_1}{1-2\sigma_2}\right)\left(\frac{1-\sigma_2}{1-\sigma_1}\right)\right] < 0 \tag{6.12}$$

since

$$\left(\frac{1-2\sigma_1}{1-2\sigma_2}\right)\left(\frac{1-\sigma_2}{1-\sigma_1}\right) < 1\,. \tag{6.13}$$

The penetration of the liner material into the joints and fractures therefore reduces the magnitude of the radial and tangential components of the displacement at the surface of the excavation and the reduction is proportional to the depth of penetration $a\varepsilon$.

Consider next the rotation of a material element at the surface of the excavation. If the shear perturbation occurs before the liner material is applied then $\sigma_1 = \sigma_2$, $E_1' = E_2'$ and $\Omega_z^{(1)}(a,\theta)$ is given by (6.4). If it occurs after the application of the liner material then $\Omega_z^{(1)}(a,\theta)$ is given by (4.28) with $\sigma_1 > \sigma_2$ and $E_1' > E_2'$. The magnitude of $\Omega_z^{(1)}(a,\theta)$ again depends on (6.5) but now since $\sigma_1 > \sigma_2$,

$$\frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} > 1 \tag{6.14}$$

and also

$$\frac{E_1}{E_2} > 1\,. \tag{6.15}$$

Thus if

$$\frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} > \frac{E_1}{E_2} > 1\,, \tag{6.16}$$

then the magnitude of the rotation is decreased while if

$$\frac{E_1}{E_2} > \frac{\sigma_1(1+\sigma_1)}{\sigma_2(1+\sigma_2)} > 1\,, \tag{6.17}$$

the magnitude of the rotation is increased by the application of the liner material. Condition (6.17) will apply if the Poisson ratio of the rock is not increased significantly by the penetration of the liner material. The magnitude of the change in rotation is proportional to $\varepsilon$ and therefore it increases linearly with the depth of penetration of the liner material.

Consider next the hoop stress at the surface of the excavation. If the shear perturbation occurs before the liner material is applied then $\sigma_1 = \sigma_2$, $E_1' = E_2'$ and $\tau_{\theta\theta}^{(1)}(a,\theta)$ is given by (6.10). If it occurs after the application of the liner material then $\tau_{\theta\theta}^{1}(a,\theta)$ is given by (4.25) with $\sigma_1 > \sigma_2$ and $E_1' > E_2'$. The magnitude of the hoop stress is increased by the factor $E_1'/E_2' > 1$. The larger hoop stress when the perturbation occurs demonstrates the stronger bonding of the rock mass due to the penetration of the liner material.

Lastly, consider the strain energy due to distortion at the surface of the excavation, $V_0^{(1)}(a,\theta)$. If the shear perturbation occurs before the application of the liner material then by setting $\sigma_1 = \sigma_2$ and $E_1' = E_2'$ in (4.29), (6.11) is obtained. If it occurs after the application of the liner material then $V_0^{(1)}(a,\theta)$ is given by (4.29) which is greater than (6.11) by the factor $E_1'/E_2' > 1$. The larger strain energy due to distortion when the shear perturbation takes place is due to the stronger bonding of the rock mass by the penetration of the liner material.

The general conclusions derived from the second and third models are essentially the same. If the shear perturbation occurs after the application of the liner material then the displacement at the surface of the excavation is reduced, the rotation will either increase or decrease depending on the relative magnitude of $\sigma_1(1+\sigma_1)/\sigma_2(1+\sigma_2)$ and $E_1/E_2$ and the hoop stress and strain energy due to distortion at the surface will increase, approximately independently of the depth of penetration, by a factor which is the ratio of $E'$ for fractured rock penetrated by liner material to $E'$ for fractured rock.

# 7 Conclusions

We found that a thin adhesive liner firmly bonded to the surface of the excavation reduced the magnitude of the stress, strain energy due to distortion, displacement and rotation in the rock at the interface. This support is provided without the penetration of liner material into joints and fractures. For a thin liner with thickness less than 20mm, second order terms in the thickness parameter $\varepsilon$ can be neglected. The reduction then is proportional to the thickness of the liner. This is in agreement with the numerical results of Stacey and Yu [8] who found a linear relation between support capacity and the thickness of the liner. The reduction is also proportional to $E_1'/E_2'$ and is therefore greater for a stiffer liner. The stiffer liner is better at preventing fracture and inhibiting the onset of rock movement although once rock movement has started a flexible liner will give better support [8]. The tensile stress in the liner is proportional to $E_1'/E_2'$ and therefore the stiffer liner attracts more stress which reduces the stress concentration factor in the rock. Liner stiffness, liner thickness, liner tensile strength and the firm bonding of the liner to the rock contribute to the support action provided by the liner. These mechanisms inhibit the onset of rock movement.

We also found that the penetration of the liner material into joints and cracks does have a significant effect on the rock mass. The predictions of the two models for liner penetration were essentially the same. The displacement due to the shear perturbation is reduced. The rotation can increase and the condition for this to happen is satisfied, for instance, if the Poisson ratio of the rock is not significantly reduced by cracks in the rock. If terms of order $\varepsilon^2$ can be neglected the changes to the displacement and rotation are proportional to the depth of penetration of the liner material. This again is in agreement with the numerical modelling of Stacey and Yu [8] who found that the support capacity increased approximately linearly with the depth of peneratration up to 10mm. In both models the the hoop stress and the strain energy due to distortion were increased by a factor which depends on the ratio of the Young's modulus in fractured rock penetrated by liner material to the Young's modulus of the fractured rock.

The actual magnitude of liner effecs can be small. When the thickness of the liner is 20mm and the radius of the excavation is 2m then $\varepsilon = 10^{-2}$. For example, with the stress concentration factor for shear stress,

$$\frac{\text{decrease in stress concentration factor}}{\text{stress concentration factor without liner}} = 4\frac{E_1'}{E_2'}\varepsilon\,. \tag{6.18}$$

If $4\,E_1'/E_2' \simeq 1$, then (6.18) is of order $10^{-2}$. The actual magnitude of liner penetration effects is also of the order of (6.18). If the depth of penetration is as large as 200mm then (6.18) is of order $10^{-1}$. Exceptions are the increase in magnitude of the hoop stress and strain energy due to distortion as a result of liner penetration which are zero order in $\varepsilon$ effects and therefore are more significant.

This work could be extended in several directions.

We investigated the effect of an adhesive liner and penetration of liner material into joints and cracks, on a cylindrical excavation subjected to a uniform shear at infinity. We could consider instead a uniform compression or tension at infinity. A remote compressive or tensile perturbation is more fundamental than an applied shear since the remote shear state can be constructed by superposition of appropriately orientated compression and tension states.

An adhesive liner that is firmly bonded to the rock will be subjected to the same deformations as the rock and since it may be stiffer or more brittle than the fractured rock it may fail under the deformations [7]. It may therefore be preferable for the rock-liner bond to be less firm in order to let shear occur on the interface making the deformation less localised. The effect of a uniform shear or tension at infinity on a cylindrical excavation with a liner less firmly bonded to the rock could be investigated.

Excavations are commonly supported by a system of liners which could consist of wire mesh, shotcrete and shotcrete reinforced by fibres. The support characteristics will be different for each system of liners. The performance of a liner on its own may be different from its performance as a component of a system [7]. The effect of a uniform shear or tension at infinity on a system of two liners could be investigated. There would be three regions and the same mathematical solution could be used to model a thin spray-on-liner which penetrates the joints and fractures in the rock to their full extent but the thickness of the liner is not neglected, or the penetration of liner material some distance into joints and fractures but the fractures do not extend to infinity. In the later case the three regions would be the rock mass into which liner material has penetrated, the further fractured rock mass and the rock mass beyond the full depth of fracturing.

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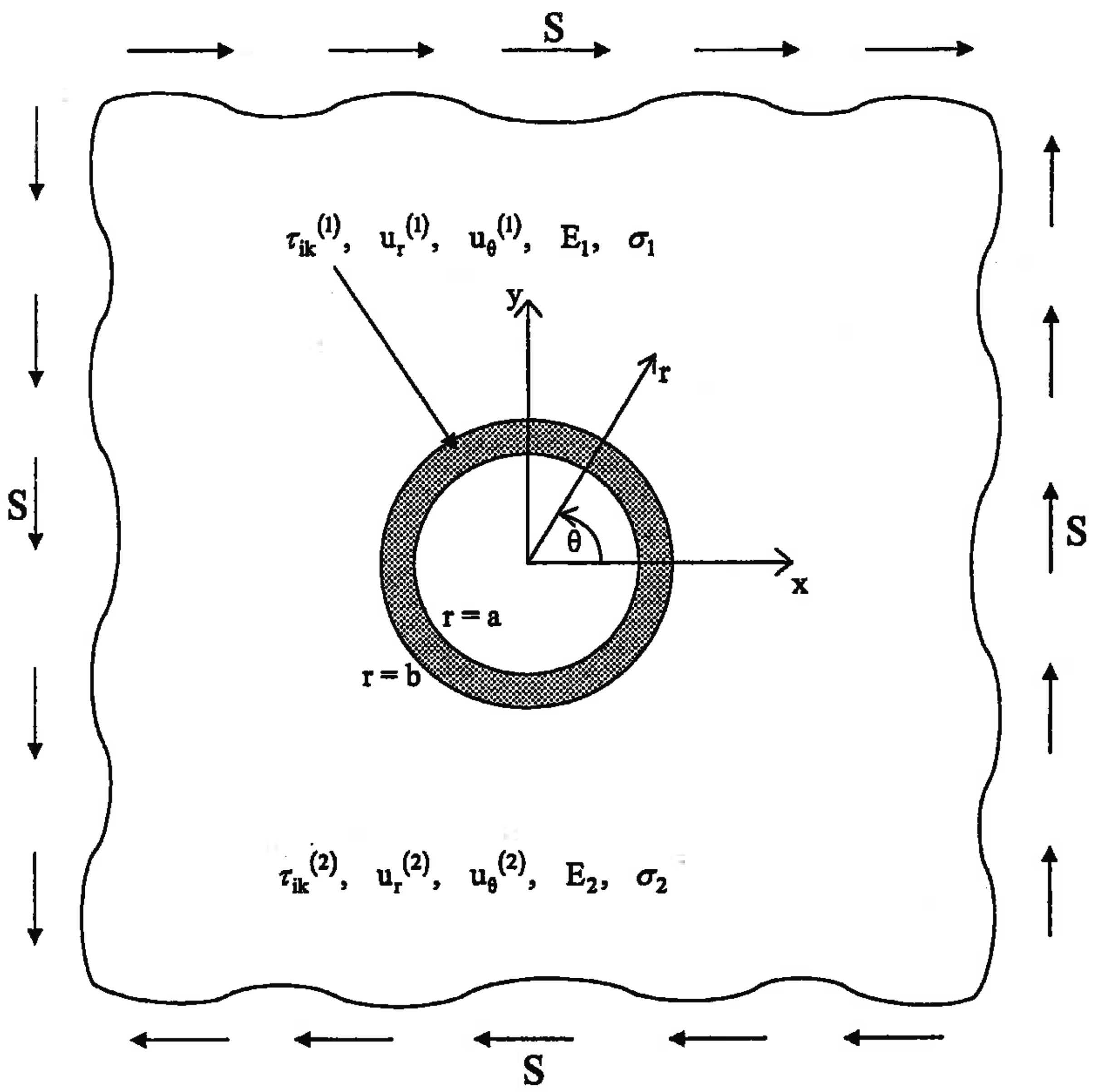


Figure 1. Cylindrical excavation in an infinite elastic medium subjected to uniform shear $S$ at infinity: region 1, $a \leq r \leq b$ and region 2, $b \leq r \leq \infty$.

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