Storage of Sugar Cane Bagasse

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Introduction

Sugar cane → Sugar cane breaks → Water added

Bagasse → Sugar Extraction

Figure: Moisture levels of 45 - 55%
Problem Description

• Stockpile as a resource
• Spontaneous combustion
• T. F. Dixon (1988)
• B. F. Gray et al (2002)

Figure: One Dimensional Model with an insulated bottom
Desired Outcomes

• Maximum height of the bagasse heap to avoid spontaneous combustion?

• Advantage in adjusting the moisture? (Usable energy per unit area)

• Advantage in pelletizing the bagasse? (Usable energy per unit area)
1D-Model formulation: B. F. Gray et. al 2001

Governing equations

\[
(\rho_b c_b + m_w X c_w) \frac{\partial U}{\partial t} = Q \rho_b Z W \exp(-E/RU) \\
+ Q_w \rho_b Z_w X W \exp(-E_w/RU)f(U) \\
+ L_v [Z_c Y - Z_e X \exp(-L_v/RU)] + \kappa \nabla^2 U, \quad (1)
\]

\[
\frac{\partial Y}{\partial t} = Z_e X \exp(-L_v/RU) - Z_c Y + D_Y \nabla^2 Y, \quad (2)
\]

\[
\frac{\partial X}{\partial t} = -Z_e X \exp(-L_v/RU) + Z_c Y, \quad (3)
\]

\[
\frac{\partial W}{\partial t} = -F \rho_b Z W \exp(-E/RU) - F \rho_b Z_w X W \exp(-E_w/RU)f(U) \\
+ D_w \nabla^2 W. \quad (4)
\]

\(U\) is temperature, \(Y\) is vapour concentration, \(X\) is liquid concentration, \(W\) is oxygen concentration.
1D-Model formulation cont’d

Boundary Conditions

At the bottom, $x = 0$, we impose the no flow condition (of heat or material)

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial Y}{\partial x} = 0, \quad \frac{\partial W}{\partial x} = 0,$$

(5)

At the top surface, $x = L$,

$$k \frac{\partial U}{\partial x} = h(U - U_a), \quad -D_Y \frac{\partial Y}{\partial x} = h_Y(Y - Y_a), \quad -D_W \frac{\partial W}{\partial x} = h_W(W - W_a),$$

(6)

Initial Conditions

$$U(x, 0) = U_0(x), \quad Y(x, 0) = Y_0(x),$$

(7)

$$X(x, 0) = X_0(x), \quad W(x, 0) = W_0(x).$$

(8)
Steady-state equations

\[ 0 = D_Y \frac{\partial^2 Y}{\partial x^2} + Z_e X \exp \left( -\frac{L_v}{RU} \right) - Z_c Y \]  
\[ \text{(9)} \]

\[ 0 = -Z_e X \exp \left( -\frac{L_v}{RU} \right) + Z_c Y \]  
\[ \text{(10)} \]

\[ Y_{xx} = 0 \Rightarrow Y_s = Y_a, \quad X_s = \frac{Z_c Y_a}{Z_e} \exp \left( \frac{L_v}{RU} \right) \]  
\[ \text{(11)} \]

\[ 0 = k \frac{\partial^2 U}{\partial x^2} + Q \rho_b Z W \exp \left( -\frac{E}{RU} \right) + Q_w \rho_b Z_w X_s W \exp \left( -\frac{E_w}{RU} \right) f(U) \]  
\[ \text{(12)} \]
1D-Model formulation cont’d

\[ 0 = D_W \frac{\partial^2 W}{\partial x^2} - F \rho_b Z W \exp \left( - \frac{E}{RU} \right) - F \rho_b Z_w X W \exp \left( - \frac{E_w}{RU} \right) f(U) \]  

(13)

If bagasse is hot (everywhere above 58C), then

\[ \frac{k}{Q} \frac{\partial^2 U}{\partial x^2} + \frac{D_W}{F} \frac{\partial^2 W}{\partial x^2} = 0 \]  

(14)

Applying boundary conditions at \( x = 0 \)

\[ \frac{k}{Q} U + \frac{D_W}{F} W = C_0 \]  

(15)

\[ \Delta Y = Y_a \quad \Delta W = W_a \quad \Delta X = \frac{Z_c Y_a}{Z_e} \exp \left( \frac{L_v}{RU_i} \right) \quad \Delta U = U_i - U_a \]
Dimensionless form

Non-dimensional model

\( \hat{t} = \frac{t}{\Delta t}, \quad \hat{x} = \frac{x}{L}, \quad \hat{U} = \frac{U - U_a}{\Delta U}, \quad \hat{Y} = \frac{Y}{\Delta Y}, \)

\( \hat{X} = \frac{X}{\Delta X}, \quad \hat{W} = \frac{W}{\Delta W}, \) (16)

Diffusion time scale is

\( \Delta t = \frac{L^2(\rho_b c_b + m_w c_w \Delta X)}{k} = \frac{L^2}{D_U}, \) (17)

The liquid equation is

\( \frac{1}{Z_e \Delta t} \exp \left( \frac{L_v}{RU_i} \right) \frac{\partial \hat{X}}{\partial \hat{t}} = -\hat{X} \exp \left( \frac{\alpha_{L_v}(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) + \hat{Y}, \) (18)

where

\( \alpha_{L_v} = \frac{L_v \Delta U}{RU_i}. \) (19)
Dimensionless form

Coefficient of LHS is $O(10^{-5})$, hence

$$
\hat{X} = \exp \left( -\frac{\alpha_{Lv}(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \hat{Y}
$$

(20)

Lose terms in heat and vapour equations

Vapour equation

$$
\kappa_Y \frac{\partial \hat{Y}}{\partial \hat{t}} = \frac{\partial^2 \hat{Y}}{\partial \hat{x}^2}, \quad \text{where} \quad \kappa_Y = \frac{L^2}{\Delta t D_Y} = O(10^{-1}).
$$

(21)

$$
(\beta_1 + \beta_2 \hat{X}) \frac{\partial \hat{U}}{\partial \hat{t}} = \frac{\partial^2 \hat{U}}{\partial \hat{x}^2} + A_E \hat{W} \exp \left( \frac{\alpha_E (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right)
$$

$$
+ A_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew} (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}),
$$

(22)
where

\[ \beta_1 = \frac{\rho_b c_b L^2}{k \Delta t} \quad \beta_2 = \frac{m_w c_w \Delta X L^2}{k \Delta t} \quad (23) \]

\[ A_E = \frac{Q \rho_b Z \Delta W L^2}{k \Delta U} \exp \left( - \frac{E}{R U_i} \right) \quad (24) \]

\[ A_{E_w} = \frac{Q_w \rho_b Z_w \Delta X \Delta W L^2}{k \Delta U} \exp \left( - \frac{E_w}{R U_i} \right), \quad (25) \]

\[ \alpha_E = \frac{E \Delta U}{R U_i}, \quad \alpha_{E_w} = \frac{E_w \Delta U}{R U_i} \quad (26) \]
Dimensionless form

The oxygen equation becomes

\[ \kappa_W \frac{\partial \hat{W}}{\partial \hat{t}} = \frac{\partial^2 \hat{W}}{\partial \hat{x}^2} - B_E \hat{W} \exp \left( \frac{\alpha_E (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \]

\[ - B_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew} (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}), \tag{27} \]

where

\[ \kappa_W = \frac{L^2}{\Delta t D_W} \quad B_E = \frac{F \rho_b Z L^2}{D_W} \exp \left( - \frac{E}{RU_i} \right) \tag{28} \]

\[ B_{Ew} = \frac{F \rho_b Z_w \Delta X L^2}{D_w} \exp \left( - \frac{E_w}{RU_i} \right). \tag{29} \]
Dimensionless form

**Boundary conditions**

At \( \hat{x} = 0 \):
\[
\frac{\partial \hat{U}}{\partial \hat{x}} = 0, \quad \frac{\partial \hat{Y}}{\partial \hat{x}} = 0, \quad \frac{\partial \hat{W}}{\partial \hat{x}} = 0, \quad \text{at} \ \hat{x} = 0,
\]

(30)

At \( \hat{x} = 1 \):
\[
- \frac{\partial \hat{U}}{\partial \hat{x}} = \gamma \hat{U}, \quad - \frac{\partial \hat{Y}}{\partial \hat{x}} = \gamma_Y (\hat{Y} - 1), \quad - \frac{\partial \hat{W}}{\partial \hat{x}} = \gamma_W (\hat{W} - 1),
\]

(31)

where
\[
\gamma = \frac{hL}{k}, \quad \gamma_Y = \frac{hYL}{D_Y}, \quad \gamma_W = \frac{hWL}{D_W}.
\]

(32)

Note \( \gamma = O(10) \), \( \gamma_Y = \gamma_W = O(10^5) \) so we may simplify the boundary conditions \( \hat{Y} = \hat{W} = 1 \) at \( \hat{x} = 1 \). The initial conditions are
\[
U = U_0, \quad Y = Y_0, \quad W = W_0, \quad \text{at} \ t = 0
\]

(33)
Discussion

Simplest model

Steady-state temperature

\[ 0 = \frac{\partial^2 \hat{U}}{\partial \hat{x}^2} + A \exp \left( \frac{\alpha(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \] (34)

This is standard form, but leads to very small piles
Discussion

What happens when the density is not assumed constant?
Pseudo steady-state

\(\kappa_W, \kappa_Y \) small

\[
\hat{X} = \exp \left( -\frac{\alpha_L (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \hat{Y} \quad \hat{Y} = 1
\]  

(35)

\[
(\beta_1 + \beta_2 \hat{X}) \frac{\partial \hat{U}}{\partial \hat{t}} = \frac{\partial^2 \hat{U}}{\partial \hat{x}^2} + A_E \hat{W} \exp \left( \frac{\alpha_E (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \\
+ A_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew} (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}),
\]  

(36)

\[
0 = \frac{\partial^2 \hat{W}}{\partial \hat{x}^2} - B_E \hat{W} \exp \left( \frac{\alpha_E (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \\
- B_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew} (\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}),
\]  

(37)
Discussion

Almost full problem

\[ \hat{X} = \exp \left( -\frac{\alpha_{Lv}(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \hat{Y} \]  

(38)

\[ \kappa_Y \frac{\partial \hat{Y}}{\partial \hat{t}} = \frac{\partial^2 \hat{Y}}{\partial \hat{x}^2} , \]  

(39)

\[ (\beta_1 + \beta_2 \hat{X}) \frac{\partial \hat{U}}{\partial \hat{t}} = \frac{\partial^2 \hat{U}}{\partial \hat{x}^2} + A_E \hat{W} \exp \left( \frac{\alpha_E(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \]

\[ + A_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew}(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}) , \]  

(40)

\[ \kappa_W \frac{\partial \hat{W}}{\partial \hat{t}} = \frac{\partial^2 \hat{W}}{\partial \hat{x}^2} - B_E \hat{W} \exp \left( \frac{\alpha_E(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) \]

\[ - B_{Ew} \hat{X} \hat{W} \exp \left( \frac{\alpha_{Ew}(\hat{U} - 1)}{U_a + \Delta U \hat{U}} \right) f(\hat{U}) , \]  

(41)
Note, insulated bottom and 100% humidity. Pile height increases with lower humidity.
Typical evolution of temperature
Often appears piles can be very large without ignition but ...

What if it rains?
Puzzle: why do apparently stable heaps ignite after getting soaked?

— wet reaction is fast, but turns off for temperatures above 58 °C,
— dry reaction is slower
So, near centre the bagasse dries out and starts to heat above the 58 °C limit. We imagine two steady states:

**inner**: hot and dry, insulated inner end, at 58 °C at interface
**outer**: warm and wet, 58 °C at interface, cooling condition at surface

Stefan problem with moving boundary
Ignition model — diagram

Please use conformal mapping to imagine this as a square with a hot yellow and a warm red band . . .

We have a good handle on equations for the steady states, but haven’t got a formulation for the velocity of the moving interface.
Conclusion

- We have a model for temperature evolution in bagasse piles - can be made simpler.

Under normal conditions, the pile does not burn, but adding water can then cause ignition. Future work will constitute consideration of a more realistic boundary condition at the bottom, a 2D model with heat loss at the sides, and comparing the full system to simplified models.
Conclusion

- We have a model for temperature evolution in bagasse piles - can be made simpler.
- Steady-state models should be sufficient - to provide bifurcation diagram.

- For any ambient conditions we can cause ignition, by making the pile sufficiently large.
- Under normal conditions pile does not burn, but adding water can then cause ignition.
- We have looked at a worst case scenario - insulated bottom, no heat loss at sides. Model can be improved.
- Future work will constitute consideration of a more realistic boundary condition at the bottom, 2D model with heat loss at the sides; compare full system to simplified models.
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References

