VORTEX LATTICE METHODS FOR HYDROFOILS

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1 Introduction

This report follows on from the executive summary and provides additional information about work carried out on the hydrofoil problem by the Study Group. The nature of the technical mathematical difficulties that arose will be described and ways of overcoming these difficulties will be presented by examining analogous situations.

A boat moving through the water experiences (hull) drag due to the generation of surface waves and due to viscous dissipation. The hydrofoil is essentially a wing placed underneath a boat which lifts the moving boat out of the water and thus greatly reduces the drag, enabling the boat to travel faster. A photograph of a hydrofoil is presented in Figure 1.

The vortex lattice method is a standard procedure for determining the fluid flow around an object (such as an airfoil) but additional technical difficulties arise in the hydrofoil case because of the presence of the water surface; indeed determining the water waves generated by the hydrofoil is the 'essence' of the problem. Gerrie Thiart has used the vortex lattice method to obtain practical numerical results for the drag

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Figure 1: A Hydrofoil.

acting on the hydrofoil, but the procedure he uses fails at high speeds as the hydrofoil moves near to the water surface. The failure displays itself in the form of divergent or nearly-divergent integrals that need to be evaluated during the computation process, and may be associated with a failure of mathematical technique, or may be due to modelling difficulties, depending on circumstances. It was not clear to the group how Gerrie Thiart currently handles the integration difficulties; it may be that he has already overcome some of the issues highlighted here.

1.1 The vortex lattice method

Historically, the vortex lattice method was one of the first computational fluid mechanics methods used, going back as late as the 1940's at the Aeronautical Research Council in the United Kingdom. The method assumes inviscid, incompressible, irrotational flow and is based on Prandtl "lifting line" theory. The method has been successfully used to predict the effect of airfoil shape on lift and drag and as such is used for aircraft design.

1.2 General theory

The velocity field $\boldsymbol{q}$ associated with an incompressible irrotational inviscid fluid flow can be described in terms of a velocity potential ϕ defined by

$$\boldsymbol{q} = \nabla\phi,$$

and this description ensures the momentum balance equation is satisfied. Imposing the continuity condition then requires that

$$\nabla^2\phi = 0,$$

so the problem reduces to one of solving Laplace's equation with suitable boundary conditions. One can do this by using a linear combination of elementary Laplace equation solutions. The elementary Laplace equation solutions used are the source and vortex solutions, and using these one can show that the general solution for the two-dimensional steady flow past a simple airfoil of length L with free stream velocity of $U\hat{\boldsymbol{e}}_x$ is given by

$$\phi = Ux + \int_0^L f(t)\log\left((x-t)^2 + y^2\right)dt + \int_0^L g(t)\arctan\left(\frac{y-t}{x}\right)dt\ ,$$

where $f(t)$ is the distribution of sources along the airfoil (providing displacement but not lift), and $g(t)$ is the distribution of vortices (providing lift but not displacement). The f and g need to be chosen so that there is no flow normal to the airfoil. The results extend to hydrofoils (consisting of a combination of simple airfoils, as illustrate in Figure 2), and if the hydrofoil is thin then no source terms are required, so it is now a matter of determining the vortex distribution $\Gamma(x, y, z)$ over the hydrofoil's surface that ensures there is no flow through this surface and also additional conditions need to be satisfied on the surface of the water, as discussed later. To do this one discretizes the wing surface into a sequence of panels with associated vortex strength Γ_{ij} and (numerically) determines the Γ_{ij}'s required to approximately satisfy the required boundary conditions, see Figure 2. Imposing these conditions leads to a set of algebraic equations for the Γ_{ij}'s. Using these Γ'_{ij}s the resultant pressure distribution around the hydrofoil can then be directly determined and lift and drag results obtained by integration.

1.3 The free surface boundary conditions

A major difficulty is that the location of the 'free' water surface needs to be determined as part of the solution process. If the surface is located at $z = \beta(x, y, t)$ then

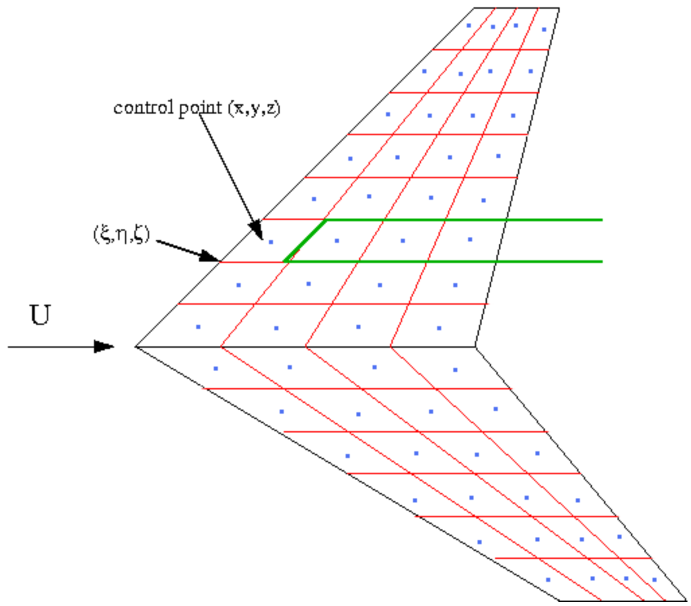

Figure 2: Discretizing the hydrofoil surface.

the nonlinear boundary conditions

$$\phi_z = \beta_t + \beta_x \phi_x + \beta_y \phi_y \qquad \text{at} \quad z = \beta(x,y,t)\ , \tag{1}$$

$$\tfrac{1}{2}\rho(\phi_x^2 + \phi_y^2 + \phi_z^2) + \rho\beta g = 0 \quad \text{at} \quad z = \beta(x,y,t)\ , \tag{2}$$

$$\phi \to 0 \qquad \text{as} \qquad z \to -\infty\ , \tag{3}$$

need to be imposed. There is no hope of a nice simple solution for the fully nonlinear problem, however by linearizing the boundary conditions and applying them on the unperturbed water surface one can obtain a tractable problem. This gives

$$\phi_z = \beta_t + U\beta_x \qquad \text{at} \quad z = 0\ , \tag{4}$$

$$\tfrac{1}{2}\rho U^2 + \rho\beta g = 0 \quad \text{at} \quad z = 0\ , \tag{5}$$

$$\phi \to 0 \qquad \text{as} \qquad z \to -\infty. \tag{6}$$

The equations for Γ_{ij} now reduce to linear algebraic equations which can be inverted. This is the procedure Gerrie Thiart used, however, difficulties arise in that terms in the solution matrix involve integrals (associated with determining the velocity on the airfoil) that diverge for small hydrofoil depths. Of course the linearization procedure requires that waves created on the surface are small which will not be the case as the hydrofoil gets closer to the water surface so inevitably the approach will fail for small enough depths. However, it may be that the range of validity of the results can be extended by suitably handling the offending integrals.

2 The problem integrals

The three integrals that need to be handled carefully and may cause problems are given by:

$$u_\omega = -\frac{\Gamma_{ij}}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{Im}\Big[f\big(\nu,\Delta\xi,\Delta\eta,\Delta\zeta\big)\Big[J\big(\nu,\xi_r,\nu_r,\zeta_r\big) - J\big(\nu,\xi_l,\nu_l,\zeta_l\big)\Big]\cos\,\nu\,d\nu, \quad (7)$$

$$v_\omega = -\frac{\Gamma_{ij}}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{Im}\Big[f\big(\nu,\Delta\xi,\Delta\eta,\Delta\zeta\big)\Big[J\big(\nu,\xi_r,\nu_r,\zeta_r\big) - J\big(\nu,\xi_l,\nu_l,\zeta_l\big)\Big]\sin\,\nu\,d\nu, \quad (8)$$

$$w_\omega = \frac{\Gamma_{ij}}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{Re}\Big[f\big(\nu,\Delta\xi,\Delta\eta,\Delta\zeta\big)\Big[J\big(\nu,\xi_r,\nu_r,\zeta_r\big) - J\big(\nu,\xi_l,\nu_l,\zeta_l\big)\Big]\,d\nu, \quad (9)$$

where

$$\begin{aligned} J(\nu,\Delta\xi,\Delta\eta,\Delta\zeta) = {}& -2i\,H(\omega)\kappa_\nu \exp\left(\kappa_\nu(z+\zeta)+i\kappa_\nu\omega\right) \\ & +\frac{1}{\pi}\left[\frac{1}{(z+\zeta)+i\omega} - \kappa_\nu\int_0^\infty \frac{e^{-t}dt}{t+\kappa_\nu((z+\zeta)+i\omega))}\right], \end{aligned} \quad (10)$$

$$f(\nu,\Delta\xi,\Delta\eta,\Delta\zeta) = \frac{\Delta\eta\ \sec\,\nu + \imath\Delta\zeta\tan\nu}{\Delta\xi\cos\nu+\Delta\eta\sin\nu+\imath\Delta\zeta}\ , \quad (11)$$

$$\kappa_\nu = \frac{g}{U^2}\ \sec^2\nu, \quad \omega = (x-\xi)\cos\nu + (y-\eta)\sin\nu$$

and H is the Heaviside function.

2.1 Problem integral 1

The first difficulties that we shall discuss come from the function

$$f(\nu,\Delta\xi,\Delta\eta,\Delta\zeta) = \frac{\Delta\eta\sec\nu + \imath\Delta\zeta\tan\nu}{\Delta\xi\cos\nu+\Delta\eta\sin\nu+\imath\Delta\zeta}.$$

Assume that for simplicity $\Delta\zeta = 0$ (flat hydrofoil). Then (for example)

$$u_\omega \propto \int_{-\pi/2}^{\pi/2} \frac{J_r - J_l}{\sin(\nu - \delta)}\, d\nu$$

where

$$\tan\delta = -\frac{\Delta\xi}{\Delta\eta}.$$

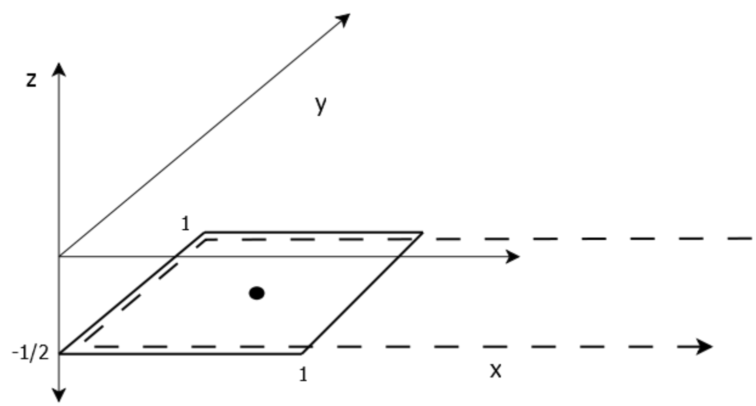

Figure 3: A grid for calculating u_ω

This exists only as a Cauchy Principal Value (CPV) integral and MUST be evaluated by suitable methods. Standard numerical integration methods (for example, Gaussian quadrature) will not, in general, cope with CPV integrals. To give a concrete illustrative example, let us take the case where we consider a flat non-tapered hydrofoil and use a one by one grid to calculate the integrals as in Figure 3. The term involving J_r behaves as shown in Figure 4 which shows a typical "Cauchy"-type singularity behaviour where $\cos\nu + \sin\nu = 0$, that is at $\nu = -\pi/4$. (The integrand behaves like $J_r(-\pi/4)/(\nu + \pi/4)$ and so the integral does not exist in a normal sense close to $-\pi/4$.) Note also the presence of the Heaviside step function at $\nu = 0.8$ which causes no problems at all because the integral is defined near to this point in a normal sense.

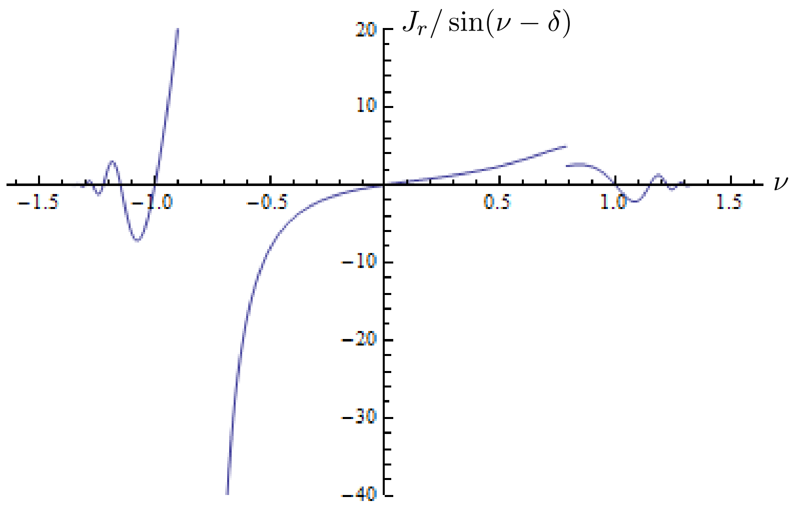

Figure 4: The $J_r(\nu)$ function.

Numerical evaluation of CPV integrals

The normal tactic is to approximate the integrand using a spectral method (for example, splines) and use "exact integration" (approximate the function UNDER the integral sign and do all integrals in closed form) to ensure that the collocation points do not bump into the singularity as the mesh is refined.
Problem Integral 1: Use the trapezium rule to evaluate

$$I_1 = \fint_0^1 \frac{t^4}{t-\frac{1}{3}}\,dt = \frac{49}{108} + \frac{1}{81}\log 2 \sim 0.4622610763$$

using a regular mesh of N intervals that does not bump into the point $t=\frac{1}{3}$. The results obtained using the trapezium rule (with different discretizations), together with the calculation using spectral methods, are displayed in Figure 5. Note that the trapezium method fails in a way that is not improved by taking more intervals; biases are introduced close to the singularity.

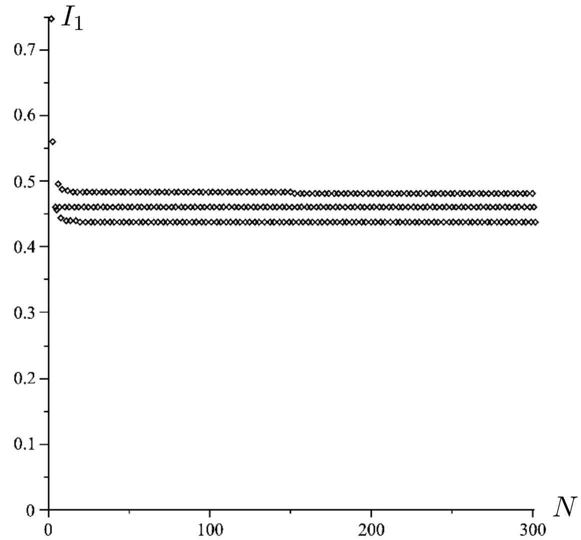

Figure 5: Numerical evaluation of I_1 using the spectral method (middle set of points). The upper and lower points correspond to $N = 1 \bmod 3$ and $N = 2 \bmod 3$.

2.2 Problem integral 2

Another severe difficulty arises from the integral term appearing in the function J. The third term in $J(\nu, \xi, \eta, \zeta))$ is of the form

$$\kappa_\nu \int_0^\infty \frac{e^{-t}}{t + \kappa_\nu((z+\zeta) + i\omega)}\, dt,$$

which in the simple case of a 1×1 grid on a flat, non-tapered 1×1 hydrofoil becomes

$$\kappa_0 \sec^2 \nu \int_0^\infty \frac{e^{-t}}{t + \kappa_0 \sec^2 \nu \left(2z + \frac{1}{2} i(\cos\nu + \sin\nu)\right)}\, dt.$$

Thus there will be a term in u_ω of the form

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 \nu}{\cos\nu + \sin\nu} \mathrm{Im}\left[\int_0^\infty \frac{e^{-t}}{t + \kappa_0 \sec^2\nu\left(2z + \frac{1}{2}i(\cos\nu + \sin\nu)\right)} dt\right] d\nu \ .$$

(Note that the $\sec^2\nu$ term in the outer integral is saved by the one in the inner integral as $\nu \to \pm\pi/2$). Multiplying by the complex conjugate we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 \nu}{\cos\nu + \sin\nu} \int_0^\infty \frac{-e^{-t}\left(\frac{1}{2}\kappa_0 \sec^2\nu(\cos\nu + \sin\nu)\right)}{(t + 2z\kappa_0 \sec^2\nu)^2 + \frac{1}{4}\kappa_0^2 \sec^4\nu\,(\cos\nu + \sin\nu)^2} dt\, d\nu \ .$$

Though the troublesome $\cos\nu + \sin\nu$ terms cancel, the denominator in the t-integral is 0 exactly when $\nu = -\frac{\pi}{4}$ *and* $t = -4z\kappa_0$. Since $z < 0$ this will inevitably happen, giving a *hypersingular* integral and severe numerical problems.

It is useful to note that the rearrangement

$$J = -i\kappa_\nu \exp(\kappa_\nu(z + \zeta) + i\kappa_\nu\omega) + \frac{1}{\pi} ⨍_0^\infty \frac{\kappa}{\kappa_\nu - \kappa} \exp(\kappa(z + \zeta) + \kappa\omega) d\kappa$$

avoids the hypersingular integral, replacing it with a less severe CPV integral together with the one discussed previously which are *not* simultaneously singular. The Giesing and Smith simplification of J is *not* the correct way to address the problem in that the resulting integrals obtained, although simpler in form, are *more* singular than needed.

2.3 Problem integral 3

Finally, the first term in $J(\nu, \xi, \eta, \zeta)$ is of the form

$$-2iH(\omega)\kappa_\nu \exp\left(\kappa_\nu (z + \zeta) + i\kappa_\nu\omega\right) \ .$$

The exponential part of this term becomes highly oscillatory under the right conditions, which correspond to the hydrofoil being moved close to the surface of the water. As an illustrative example, for a flat hydrofoil, the exponential can be written as

$$E = \exp\left(\kappa_0 \sec^2\nu(2z + i\omega)\right) .$$

Since $z < 0$, when ν is within a distance ϵ of $\pm\pi/2$, we have

$$E \sim \exp\left(\frac{\kappa_0}{\epsilon^2 + O(\epsilon^4)}\big(- 2|z| + i\omega\big)\right) .$$

The complex term in the exponential causes high frequency oscillations as $\epsilon \to 0$, but the real term damps them out. For the oscillation to be "sufficiently" damped we require (by numerical experiment - more could be done with more time)

$$2|z| \approx\geq \frac{\omega}{10} .$$

3 Allowable distance from fluid surface:

Recall that the linear theory will eventually fail for small enough hydrofoil depths. Here we estimate the validity range for the linear theory. It should be noted that the complete non-linear theory involves issues that are not yet fully understood, let alone quantifiable. Waves that form on the surface may break or may propagate and this will strongly effect the hydrofoil's performance. Furthermore seemingly minor design 'details' can effect the outcome in this operational range. These issues may or may not be practically important, but in any case it is important to know when the linear results can be believed.

Now consider a hydrofoil (span $2l$, chord α) with the centre of the leading edge at $(x, y) = (0, 0)$. We divide the hydrofoil into a $2M \times N$ grid, and our task is to determine the largest possible value of $\omega = (x - \xi)\cos\nu + (y - \eta)\sin\nu$. This will allow us to determine a bound for $|z|$ and thus tell us how close to the surface we can bring the hydrofoil.

Note that $\max(\omega)$ occurs when calculating the value induced at the top corner control point by the vortex segment starting at the origin. This maximal value of ω was found to be given by

$$\omega = \left(\alpha - \frac{\alpha}{2N}\right)\cos\nu + \left(l - \frac{l}{2m}\right)\sin\nu.$$

The maximum value of ω is

$$\omega_{max} = \sqrt{\left(\alpha - \frac{\alpha}{2N}\right)^2 + \left(l - \frac{l}{2m}\right)^2},$$

so we have

$$|z| \approx\geq \frac{1}{20}\sqrt{\left(c - \frac{c}{2N}\right)^2 + \left(l - \frac{l}{2M}\right)^2} . \qquad (12)$$

Placing the hydrofoil closer to the surface causes the integrands in (7,8,9) to become highly oscillatory corresponding to the breakdown of the linearised boundary condition, as illustrated in Figure 6. One might be able to analytically evaluate such integrals for an extended parameter range but any such gains are limited.

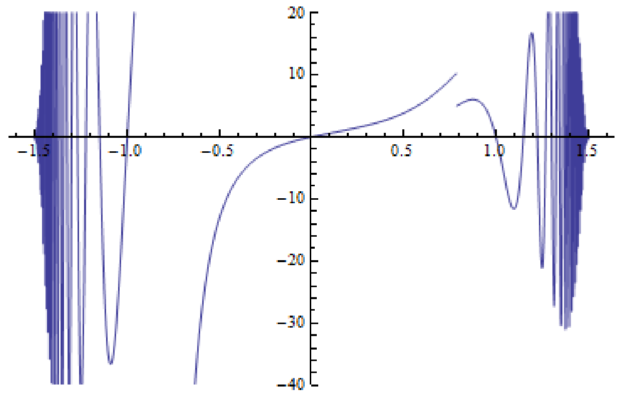

Figure 6: Typical integrand behaviour associated with (7,8,9) when the airfoil moves close to the surface.

4 Final conclusions

- It is absolutely necessary to use a suitable method to calculate the CPV integrals and "spectral" methods are highly recommended.
- The Giesing and Smith simplification of J is not the useful way to proceed. The less convenient-looking original version is much to be preferred in that this has CPV integrals, but no hypersingular integrals.
- Use the criterion (12) to determine the applicability range of the linear theory.
- The linear theory *will* fail for sufficiently small hydrofoil depths, and there is no known simple fix to cover this range; possibly detailed experiments would be necessary.

References

[1] Thiart, G.D. Vortex lattice method for a straight hydrofoil near a free surface. *International Shipbuilding Progress*, **44** (1997), 5-26.

[2] Thiart, G.D. Generalized vortex lattice method for prediction of hydrofoil characteristics. Private communication, 1999.